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A perturbative analysis of the time-envelope approximation in strong Langmuir turbulence

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Abstract

We investigate a nonlinear set of coupled-wave equations describing the inertial regime of the strong Langmuir turbulence, namely

$$\frac{1}{\omega^2}\frac{\partial^2 E}{\partial t^2} - 2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = -nE,$$
$$\frac{1}{c^2}\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2,$$

which differs from the usual Zakharov equations by the inclusion in the first equation for E of a second time-derivative, multiplied by the parameter $1/\omega^2$ that vanishes under the so-called time-envelope approximation $\omega^2 \to +\infty$. From these perturbed Zakharov equations, it is shown that the latter limit is not compatible with a strongly dominant ion inertia corresponding to the formal case $c^2 \to 0$. In the opposite case, i.e. as c^2 remains of order unity, the local-in-time Cauchy problem attached to the above equations is solved and the limit $\omega^2 \to +\infty$ is detailed for a fixed value of c^2. Under some specific initial data, the solution E is proved to blow up at least in an infinite time provided that ω lies below a threshold value. When this condition is not fulfilled, the global existence of the solution set (E, n) is finally restored in a one-dimensional space.

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1. Introduction

1.1. The physical problem

For two decades, the strong Langmuir turbulence (SLT) has often been emphasized as being a natural way for saturating the sudden growth of electron plasma waves interacting with ion density fluctuations [1] : the typical scenario for damping "energetic" plasma waves (also called Langmuir waves) is that these high-frequency (HF) waves, carried by electron oscillations, couple with low-frequency (LF) ion fluctuations giving rise in the medium to localized cavities inside which Langmuir wave energy is trapped. Remaining coupled to the ions by the so-called ponderomotive force that locally expels the electrons out of the plasma regions where strong electron oscillations develop, the HF Langmuir waves self-contract in the so-called inertial regime of the SLT where no dissipation takes place, and generate intense and spiky electric fields in the medium. When their typical size becomes of the order of a few Debye lengths λ_D, i.e. as the typical wavenumber k of the Langmuir waves increases such as $k\lambda_D \to 0.3$ for a Maxwellian electron distribution function, Landau damping becomes efficient and burns out the Langmuir field (the Debye radius classically corresponds to the elementary radius of the screening cloud formed by electrons surrounding an isolated ion). The key mechanism of this physical phenomenon is governed by the well-known Zakharov equations [2] that describe the slowly varying motions of the complex-valued envelope of the Langmuir electric field which nonlinearly interacts with large-scale density fluctuations through the ponderomotive force. Their standard derivation follows from combining the Maxwell equations describing the longitudinal component of the Langmuir electrostatic field with the fluid equations related to the dynamical motions of the electron/ion densities hereafter denoted by N_e and N_i. By doing so, and when assuming a globally neutral charged medium with $N_e = ZN_i$ for a plasma of charge number Z, the dynamical equation governing the scalar potential ϕ of the longitudinal field ($\mathbf{E} = -\nabla\phi$) is found to read as

$$\Delta\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2 - 3v_{th}^2\Delta\right)\phi = -\omega_{pe}^2 \mathrm{div}\frac{\partial}{\partial t}\left(\frac{\delta n}{N_0}\int_0^t \nabla\phi(u)\,\mathrm{d}u\right). \quad (1.1)$$

Here, $\omega_{pe}[N_0] = \sqrt{q_e^2 N_0/\epsilon_0 m_e}$ is the electron plasma frequency where q_e, m_e, ϵ_0 and N_0, respectively denote the electric charge and mass, the vacuum dielectric constant and the background electron density; $v_{th} = \lambda_D\omega_{pe} \simeq \sqrt{T_e/m_e}$ is the electron thermal velocity, and δn represents a small fluctuation of the nonuniform density $N_e = N_0(1 + \delta n/N_0)$ satisfying $\delta n/N_0 << 1$. An alternative way of recovering (1.1) is to take formally the inverse Fourier transform of the characteristic dispersion relation of Langmuir waves with frequency $\tilde{\omega}$, namely [3]

$$\tilde{\omega}^2 = \omega_{pe}^2[N_0 + \delta n](1 + 3k^2\lambda_D^2), \quad (1.2)$$

where the wavenumber k usually obeys the condition $k^2\lambda_D^2 << 1$ when regarding large-scale fluctuations on which the Landau damping remains inefficient. In (1.2), the perturbation δn contains in principle a low-frequency contribution δn_{LF} and a high-frequency one δn_{HF} carried by the electron oscillations, such that it simply consists in the amount of both components : $\delta n = \delta n_{LF} + \delta n_{HF}$. The latter component of the perturbation is directly integrated from the Poisson equation $\Delta\phi = -q_e\delta n_{HF}/\epsilon_0$, and it only contributes in the right-hand side of (1.1) to a nonresonant term which does not beat at the plasma electron frequency ω_{pe} when one introduces into this equation of motion the envelope substitution

$$\phi(\mathbf{x}, t) = \frac{1}{2}(\psi(\mathbf{x}, t)\exp(-\mathrm{i}\omega_{pe}t) + \mathrm{c.c.}). \quad (1.3)$$

When retaining only the terms of (1.1) beating at ω_{pe} [more precisely, when averaging (1.1) times $e^{i\omega_{pe}t}$ with $\partial_t \psi << \omega_{pe}\psi$ over a period $2\pi/\omega_{pe}$], the potential contribution induced by δn_{HF} can therefore be cancelled, so that only the low-frequency part of the perturbation δn_{LF} plays an effective role. Thus, Eq. (1.1) reduces to the simplified equation

$$\Delta\left(\frac{\partial^2}{\partial t^2} - 2i\omega_{pe}\frac{\partial}{\partial t} - 3v_{th}^2\Delta\right)\psi = -\omega_{pe}^2 \operatorname{div}\left(\frac{\delta n_{LF}}{N_0}\nabla\psi\right), \tag{1.4}$$

where the second time-derivative is assumed to be small (this follows from the average assumption $\partial_t\psi << \omega_{pe}\psi$), but consists in a *resonant* contribution which can be kept. Furthermore, as the short-scale HF fluctuations have been neglected, the perturbation $\delta n \approx \delta n_{LF}$ corresponds to either large-scale electron or ion motions by virtue of the quasi-neutrality assumption. Those motions are described by a sound-wave equation including the ponderomotive force $\mathbf{F}_{pm} = -\frac{m_e}{2}\nabla\langle \mathbf{v}_{HF}^2\rangle$, which expresses as

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\Delta\right)\frac{\delta n}{N_0} = \frac{Zm_e}{2m_i}\Delta\langle \mathbf{v}_{HF}^2\rangle. \tag{1.5}$$

Here, m_i is the ion mass, $\mathbf{v}_{HF}$ denotes the high-frequency electron velocity related to the electric field by the dynamical equation $\partial_t \mathbf{v}_{HF} = (q_e\mathbf{E}/m_e)$ and the brackets $\langle . \rangle$ represent a time-averaging over a $2\pi\omega_{pe}^{-1}$ period. Using the previous assumptions together with the envelope substitution (1.3), $\langle \mathbf{v}_{HF}^2\rangle$ can easily be checked to reduce at the main order to $\langle \mathbf{v}_{HF}^2\rangle \cong (q_e^2/2\omega_{pe}^2 m_e^2)|\nabla\psi|^2$: by "main order", it must be understood that the previous estimate is true up to some small corrections in $o((\tilde\omega - \omega_{pe})/\omega_{pe})$ which may be disregarded when considering the first-order contribution of $\delta n/N_0$ as the Langmuir wave frequency remains close to ω_{pe}. For the sake of simplicity, we select the longitudinal component of the Langmuir electric field [3], and setting $\nabla \equiv (\mathbf{k}/k)\cdot\nabla$ for notational convenience, we introduce $\tilde E = -\nabla\psi$ as being the scalar envelope of this field, so that the vectorial system (1.4) and (1.5) can finally be approached by the following scalar model:

$$\frac{\partial^2 \tilde E}{\partial t^2} - 2i\omega_{pe}\frac{\partial \tilde E}{\partial t} - 3v_{th}^2\Delta\tilde E = -\omega_{pe}^2\frac{\delta n}{N_0}\tilde E, \tag{1.6}$$

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\Delta\right)\frac{\delta n}{N_0} = \frac{Z\epsilon_0}{4m_i N_0}\Delta|\tilde E|^2. \tag{1.7}$$

Resulting from an integration of (1.4) over space, Eqs. (1.6) and (1.7) are defined up to a spatially uniform integration constant that may represent an external "pump" driver which is here disregarded as we deal with a "free" plasma turbulence. In this case, the above system restores the well-known Zakharov equations stated under the so-called time-envelope approximation $\partial_t^2\tilde E << 2i\omega_{pe}\partial_t\tilde E$, i.e. when the second time-derivative of $\tilde E$ is ignored in (1.6). This paper is devoted to the previous approximation. In order to investigate this problem properly, we now need to transform (1.6) and (1.7) into a convenient formulation exhibiting the perturbative nature of the second time-derivative of $\tilde E$ in (1.6).

1.2. A suitable rescaling of the perturbed Zakharov equations

The aim of this Section is to re-express the perturbed Zakharov equations (1.6) and (1.7) in such a way that the second time-derivative in (1.6) plays the role of a perturbative parameter tending to zero as the time-envelope approximation $\partial_t^2\tilde E << 2i\omega_{pe}\partial_t\tilde E$ is imposed. Let us first recall that when the latter limit is a priori applied, system (1.6) and (1.7) is classically rewritten in a reduced system of units based upon some dimensionless space–time variables $t \to \omega_{pe}t$ and $\mathbf{x} \to k_D\mathbf{x}$ with $k_D = 2\pi/\lambda_D$ [1]. Here, we need to normalize the variables and fields such that all the terms occurring in (1.6) and (1.7) are of the same order, except the one in $\partial_t^2 \tilde{E}$ that is expected to vanish under the time-envelope limit. To construct such a rescaling, we introduce the plasma wave envelope frequency as defined by $\delta\omega = \tilde{\omega} - \omega_{pe}$. In the inertial regime of the SLT, the latter quantity is assumed to remain very small since k and δn satisfy $k^2\lambda_D^2 << 1$ and $\delta n/N_0 << 1$, respectively, so that $\tilde{\omega}$ remains close to ω_{pe}. Under these hypotheses, the dispersion relation (1.2) can be expanded to yield

$$\delta\omega = \frac{3}{2}\omega_{pe}k^2\lambda_D^2 + \frac{\delta n}{2N_0}\omega_{pe} \tag{1.8}$$

that determines the orders of magnitude $\delta\omega \approx \frac{3}{2}\omega_{pe}k^2\lambda_D^2 \approx (\delta n/2N_0)\omega_{pe}$ which must be satisfied by the initial data under investigation. Using these estimates, one can thus line up the temporal variable t – not with the plasma electron frequency ω_{pe} – but with the envelope frequency (1.8) that we henceforth regard as being a free parameter to be discussed later on. Accordingly with this transform $t \to \delta\omega t$, we also line up the space variables with the characteristic wavenumber of the Langmuir envelope, namely $\mathbf{x} \to k\mathbf{x}$. Together with these new space–time variables

$$t \to t' = \delta\omega t \quad (\partial_t^2 \to \delta\omega^2 \partial_t'^2), \tag{1.9}$$

$$\mathbf{x} \to \mathbf{x}' = k\mathbf{x} \quad (\Delta \to k^2\Delta'), \tag{1.10}$$

we self-consistently change the scalar densities δn and $|\tilde{E}|^2$ by means of the former estimates, namely

$$\frac{\delta n}{N_0} \to n' = \frac{\omega_{pe}}{\delta\omega}\frac{\delta n}{N_0} = \mathrm{O}(1), \tag{1.11}$$

$$|\tilde{E}|^2 \to |E'|^2 = \frac{\omega_{pe}}{\delta\omega}|\tilde{E}|^2 = \mathrm{O}(1), \tag{1.12}$$

where the notation O(1) refers to some quantities that remain independent of the ratio $\omega_{pe}/\delta\omega$. We then emphasize that for a usual laser-created plasma in controlled fusion experiments, one has $T_e >> T_i$ such as $c_s^2 \simeq ZT_e/m_i$, and eventually normalize $|\tilde{E}|^2$ in energy units $(4N_0T_e/\epsilon_0)$ to obtain the following rescaled set of perturbed equations:

$$\frac{1}{\omega^2}\frac{\partial^2 E}{\partial t^2} - 2\mathrm{i}\frac{\partial E}{\partial t} - 2\Delta E = -nE, \tag{1.13}$$

$$\frac{\alpha^2}{\omega^2}\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta|E|^2, \tag{1.14}$$

where – for the sake of clarity – the prime notation has been dropped out. In Eqs. (1.13) and (1.14), the quantity $1/\omega^2$ denotes the small parameter

$$\frac{1}{\omega^2} = \frac{\delta\omega}{\omega_{pe}} \tag{1.15}$$

that reflects the deviation of the time-enveloped Zakharov equations from the original HF wave equation (1.1). Imposing the time-envelope approximation is then nothing else but taking the limit $\omega^2 \to +\infty$ in Eq. (1.13), which thereby reduces to a singular perturbation problem. Moreover, the coefficient (α^2/ω^2) in front of the time-derivatives of (1.14) also displays an explicit dependence of the ion-sound wave frequency on the envelope frequency. By definition, this coefficient reads as

$$\frac{\alpha^2}{\omega^2} = \frac{\delta\omega^2}{c_s^2k^2}, \tag{1.16}$$

and when one makes use of the estimate $\tilde{\omega}^2 - \omega_{pe}^2 \approx 3v_{th}^2 k^2$ leading to $k^2 \simeq (2\delta\omega\omega_{pe}/3v_{th}^2)$, the constant α in (1.16) is simply related to the ratio of the thermal electron velocity to the ion-acoustic speed as follows :

$$\alpha = \sqrt{\frac{3}{2}}\frac{v_{th}}{c_s} \cong \sqrt{\frac{3}{2\mu}} \tag{1.17}$$

with

$$\mu = \frac{Zm_e}{m_i} << 1.$$

1.3. The different regimes occurring in SLT

A first consequence directly inferred from the rescaled equation (1.13) is that the usual time-envelope approximation is recovered by passing to the formal limit $\omega \to +\infty$: by "formal" limit, it is simply meant that the ratio $(\delta\omega/\omega_{pe})$ in (1.15) never strictly vanishes from a physical viewpoint, but is here viewed as becoming rather negligible in front of the remaining contributions of order unity in (1.13) and (1.14). In other words, the limit $\omega^2 \to \infty$ signifies that the envelope frequency $\delta\omega$ becomes infinitely low as compared with the basic HF electron frequency ω_{pe}. Modeling the SLT even when keeping some "residual" electron oscillations represented by the second time-derivative of (1.13) permits to refind the main dynamical regimes of large-scale cavities n evolving in a turbulent plasma, namely the subsonic and the supersonic regimes whose properties are now reviewed in the following:

(i) *The subsonic regime* usually applies to ion fluctuations whose temporal dynamics is ignored, i.e. as $\delta n/N_0$ satisfies

$$\partial_t^2 \delta n/N_0 << c_s^2 \Delta\delta n/N_0$$

in (1.7). In this ion-static (so-called "adiabatic") limit, Eq. (1.14) simplifies into $n = -|E|^2$ and (1.13) reduces to a nonlinear Schrödinger equation, the solutions of which have been recently examined in the time-envelope limit by Bergé and Colin [4]. From a physical point of view, the applicability domain of the subsonic regime concerns Langmuir envelopes coupled with ion cavities whose typical extensions in space and in time – denoted by L and T, respectively – satisfies the relation $L/T < c_s$, as deduced from the sound-wave equation (1.7) preserved in physical units. In addition to this latter estimate, we can use Eq. (1.6) to establish that under the time-envelope approximation, the respective sizes L and T of the Langmuir envelope proceed from the following ordering:

$$\frac{2}{\omega_{pe}T} \sim \frac{3\lambda_D^2}{L^2} \sim \frac{\delta n}{N_0} \tag{1.18}$$

showing that in the subsonic regime $L/T < c_s$, the cavities $\delta n/N_0$ evolve with a bounded amplitude

$$\left|\frac{\delta n}{N_0}\right| << \frac{4c_s^2}{3v_{th}^2} \approx \mu << 1.$$

We recall on this purpose that the former ordering (1.18) may be justified by performing a linear stability analysis of the time-enveloped Zakharov equations expressed in a one-dimensional (1-D) space. One can indeed search for the unstable modes that tend to break up a spatially uniform time-independent Langmuir envelope $\tilde{E}_0$ by assuming $\tilde{E} = \tilde{E}_0 + \tilde{E}_1 \exp(\gamma t)\cos(kx)$ ($\tilde{E}_1 << \tilde{E}_0$) and deduce that the latter modes are characterized by a maximum growth rate $\gamma \approx 1/T$ reached for unstable mode wavenumbers $k = 2\pi/L \approx |\delta n(\tilde{E}_0)/N_0|^{1/2}$. This standard analysis reveals the modulational instability of a long-wavelength Langmuir wave induced by low-amplitude density fluctuations. Modulational instability arises within the subsonic regime when the cavity fluctuations verify $|\delta n(\tilde{E}_0)/N_0| << \mu$, or equivalently when the density of Langmuir electrostatic energy initially satisfies the inequality $W \equiv (\epsilon_0|\tilde{E}_0|^2/4N_0T_e) << \mu$ (see, e.g. [5]). Moreover, as inferred from the above estimates (1.18), the product $k\lambda_D$ characterizing unstable Langmuir waves of typical wavenumbers $k \sim 1/L$ belongs to the range $k\lambda_D < \sqrt{\mu}$.

(ii) *The supersonic regime* corresponds to cavity motions governed by the opposite approximation

$$\partial_t^2 \delta n/N_0 >> c_s^2 \Delta \delta n/N_0$$

and develops in a turbulent medium as soon as the ion dynamics grows up dominantly as compared with the ion compression effects. By repeating the above dimensionality arguments, it follows that the latter inequality applies to space–time envelope scales L and T satisfying $L/T > c_s$. Besides, using again the estimate (1.18) yields an opposite inequality $|\delta n/N_0| >> \mu$ which means that the cavity fluctuations exhibiting a too large deepening destabilize the amplitude of Langmuir waves in the supersonic stage of the modulational instability. This stage essentially concerns some large-amplitude Langmuir fields characterized by the inequality $W >> \mu$, and its typical spectrum in k ranks among the large wavenumbers $k\lambda_D > \sqrt{\mu}$, as easily reported from the former dimensionality analysis.

Let us first note that the above characteristics of the inertial SLT are well-restored by our model (1.13) and (1.14): as previously stated, every term of (1.13) and (1.14) remains of order unity, apart from those affected by some ω-dependent coefficients that contain by themselves the nature of the turbulent regime. We can then check from (1.16) that a simple expansion of $\alpha/\omega < 1$ leads to $k^{-1}\delta\omega < c_s$ which is nothing else but the subsonic approximation $L/T < c_s$, whereas $\alpha/\omega > 1$ yields the complementary limit $L/T > c_s$. Furthermore, keeping in mind the typical order of magnitude $\omega^{-2} = (\delta\omega/\omega_{pe}) \approx (\frac{3}{2}k^2\lambda_D^2)$, one easily sees that a supersonic (subsonic) evolution of the cavities is privileged in the spectral domain $k\lambda_D > \sqrt{\mu}$ (resp. $k\lambda_D < \sqrt{\mu}$), as soon as ω^2 is ensured to be smaller (resp. larger) than the quantity $\alpha^2 = 3/(2\mu)$, i.e. when the frequency ratio α/ω satisfies $\alpha/\omega > 1$ (resp. $\alpha/\omega < 1$).

The previous analysis suggests that the parameter α^2/ω^2 in Eq. (1.14) can be tuned as being more or less than the unity for recovering the main properties of the SLT without dissipation. In addition to these reviews, a salient result follows from (1.13) and (1.14) when bearing in mind that α^2 is in fact the single physical constant of our rescaled problem. Indeed, passing crudely to the limit $\omega \to +\infty$ should imply not only the second time-derivative of the Langmuir envelope to vanish, but above all to force a subsonic evolution for the fluctuating cavities n as $\alpha^2/\omega^2 \to 0$. Inversely, the rescaled system (1.13) and (1.14) clearly displays that *the development of a too strong supersonic regime characterized by the inequality $\alpha >> \omega$ is not compatible with the time-envelope limit $\omega \to +\infty$*. As a consequence of this result, we can conclude that the time-envelope approximation does not hold any longer as soon as the temporal fluctuations of the ion cavities become drastically enhanced, i.e. as the proper frequency $\Omega \sim \alpha/\omega$ associated with the ion fluctuations exceeds too much the unity (this amounts to dealing with extreme situations concerned by the asymptotic regions of the supersonic regime $(\delta\omega/\omega_{pe}) >> 2\mu/3$, for which $\delta\omega$ could become comparable to ω_{pe}). As a matter of fact, such a situation may never occur in a real turbulent plasma, because in the contrary case, it should concern the asymptotic part of the k-spectrum lying in the range $k\lambda_D >> \sqrt{\mu}$, and therefore reaching the dissipative domain $k\lambda_D \to 0.3$ for which the "undamped" Zakharov equations – as well as their rescaled version (1.13) and (1.14) – become quite invalid.

In summary, applying simultaneously the limit $\omega^2 \to +\infty$ in both of Eqs. (1.13) and (1.14) forces not only the second time-derivative of E, but also the one of n, to disappear. As a result of the latter effect, n simplifies to just $-|E|^2$. We are, however, not able to work out this full limit, for reasons discussed in Appendix A.

Nevertheless, since $1/\omega$ never truly vanishes for a turbulent plasma medium, while the constant α^2 is quite large (typically $\alpha^2 \approx 10^3 - 10^4$), the coefficient $1/\omega^2$ may be very small and the coefficient α^2/ω^2 be much larger. It is therefore worthwhile considering the limit for which the former tends to zero, whereas the latter remains fixed. This assumption is fully justified when investigating the inertial supersonic regime of the SLT characterized by the spectral domain $\sqrt{\mu} < k\lambda_D < 0.3$. Fixing this range of k indeed amounts to regarding the limit $\frac{3}{2}(\alpha k\lambda_D)^2 \approx \alpha^2/\omega^2$ of order unity with a large value $\alpha^2 >> 1$, while $\delta\omega/\omega_{pe}$ is viewed as a sufficiently small quantity for the formal limit $\omega^2 \to +\infty$ to make sense. The main part of the forthcoming analysis will be devoted to this regime.

Although of less physical relevance, it is besides possible to consider the limits in which the coefficient ω/α tends to infinity, leaving a perturbed cubic Schrödinger equation, and afterwards to pass to the limit $\omega \to \infty$ in the latter, reducing once again to the usual nonlinear Schrödinger equation. The former limit will also be treated here, while the latter was recently justified by Bergé and Colin [4].

1.4. Setting of the problem and statement of the results

In the remaining of this paper, we will study the mathematical properties corresponding to the above physical situations that can be modeled through the nonlinear equation set

$$\begin{aligned} &\frac{1}{\omega^2}\frac{\partial^2 E}{\partial t^2} - 2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = -nE, \\ &\frac{1}{c^2}\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2, \end{aligned} \tag{1.19}$$

where the numerical factor 2 in front of the Laplacian of (1.13) has been dropped after a simple rescaling of the space variables, so that c now represents the ratio $(\omega/\sqrt{2}\alpha)$. Let us first recall what kind of results are nowadays known about this family of systems.

(i) The first case is $\omega = c = +\infty$ and leads to the cubic Schrödinger equation which has been extensively studied in the past (see for a review: [6] or [7]).

(ii) The case $\omega = +\infty$ and $c < +\infty$ corresponds to the Zakharov equations whose well-posedness of the Cauchy problem has been addressed by Sulem and Sulem [8] and Ozawa and Tsutsumi [9]. Furthermore, the limit $c \to \infty$ has been treated by Schochet and Weinstein [10]. Some partial results on time blow-up can also be found in [12,13].

(iii) As recalled in Section 1.3, the case $\omega < +\infty$ and $c = +\infty$ has been recently studied by Bergé and Colin in [4] where the asymptotics $\omega \to +\infty$ has been accurately described.

In what follows, we fix c and we investigate the limit $\omega \to +\infty$. The main results relative to this problem can be summarized as follows (see Theorems 1–3):

Let $s > \frac{1}{2}d$ and $E_0 \in H^{s+2}(\mathbb{R}^d)$, $E_1 \in H^s(\mathbb{R}^d)$, $n_0 \in H^{s+1}(\mathbb{R}^d)$, $n_1 \in H^s(\mathbb{R}^d)$ and suppose that $(|E_1|_{H^{s+1}}/\omega) \to 0$ as $\omega \to +\infty$, then there exists $T^\omega > 0$ corresponding to a unique maximal solution to

$$\frac{1}{\omega^2}\frac{\partial^2 E^\omega}{\partial t^2} - 2\mathrm{i}\frac{\partial E^\omega}{\partial t} - \Delta E^\omega = -n^\omega E^\omega,$$

$$\frac{1}{c^2}\frac{\partial^2 n^\omega}{\partial t^2} - \Delta n^\omega = \Delta |E^\omega|^2,$$

$$E^\omega(0) = E_0, \quad E^\omega_t(0) = E_1, \quad n^\omega(0) = n_0, \quad n^\omega_t(0) = n_1,$$

satisfying

$$E^\omega \in C([0, T^\omega[; H^{s+2}) \cap C^1([0, T^\omega[; H^s),$$

$$n^\omega \in C([0, T^\omega[; H^{s+1}) \cap C^1([0, T^\omega[; H^s).$$

Let E be the solution to the Zakharov equations

$$-2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = -nE,$$

$$\frac{1}{c^2}\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2,$$

$$E(0) = E_0, \quad n(0) = n_0, \quad n_t(0) = n_1,$$

and T^∞ be its existence time. Then, we have

$$\liminf_{\omega\to\infty} T^\omega \geq T^\infty,$$

together with the following limits valid as $\omega^2 \to \infty$ for all $T_1 < T^\infty$:

$$E^\omega \to E \text{ in } L^\infty(0, T_1; H^{s+2}),$$

$$E_t^\omega - E_t - \mathrm{e}^{2\mathrm{i}\omega^2 t} w(x,t) \to 0 \text{ in } L^\infty(0, T_1; H^s),$$

$$n^\omega \to n \text{ in } L^\infty(0, T_1; H^{s+1}),$$

$$n_t^\omega \to n \text{ in } L^\infty(0, T_1; H^s),$$

where w is the solution to

$$2\mathrm{i}\frac{\partial w}{\partial t} - \Delta w = -nw,$$

$$w(x,0) = -\frac{\mathrm{i}}{2}\Delta E_0 + E_1 + \frac{\mathrm{i}}{2} n_0 E_0.$$

Suppose moreover that $n_1 \in \dot{H}^{-1}(\mathbf{R}^d)$, and introduce the quantity $I(t) = \frac{1}{2}\int_{\mathbf{R}^d} |E^\omega|^2$ together with the function ψ^ω obeying the continuity relation $c^2 \Delta\psi^\omega = n_t^\omega$. If E^ω and n^ω exist for all times and if both of the invariant integrals

$$Q_\omega \equiv \int_{\mathbf{R}^d} |E^\omega|^2 - \frac{1}{\omega^2}\mathrm{Im}\int_{\mathbf{R}^d} \frac{\partial E^\omega}{\partial t}\bar{E}^\omega,$$

$$\mathcal{E}_\omega \equiv \int_{\mathbf{R}^d} \left(\frac{1}{\omega^2}\left|\frac{\partial E^\omega}{\partial t}\right|^2 + |\nabla E^\omega|^2 + n^\omega |E^\omega|^2 + \frac{c^2}{2}(\nabla\psi^\omega)^2 + \frac{1}{2}(n^\omega)^2\right)$$

satisfy one of the two conditions

(C1) $\quad \mathcal{E}_\omega + 2\omega^2 Q_\omega < 0$

(C2) $\quad \mathcal{E}_\omega + 2\omega^2 Q_\omega = 0, \quad \dot{I}(0) > 0,$

then,

$$\lim_{t\to\infty} |E^\omega|_{L^2} = +\infty.$$

To solve the above-summarized Cauchy problem, the appropriate method is to prove the existence of a fixed-point of an equation obtained after some suitable transformations of the initial system. The outline of the article is the following:

In Section 2, we establish a well-posedness result for the system (1.19) whose solution is found to be bounded independently of ω and characterized by an existence time uniform with respect to ω.

In Section 3, we perform the limit process $\omega \to \infty$.

In Section 4, we prove that a peculiar class of solutions to (1.19) blow up at least at an infinite time when ω lies below a critical value. In the contrary case, i.e. if ω is large enough, we show that for fixed initial data in space dimension 1, the solutions are global in time.

We finally devote an appendix in order to explain why we do not know how to conclude in the case when simultaneously both the limits $\omega \to \infty$ and $c \to \infty$ are taken. We state a result when ω is fixed and $c \to \infty$ and we give a brief sketch of proof, showing that the transformations introduced in Section 2 apply in this simpler case.

Note that, provided a nonlocal operator of the form $[\mathrm{grad}(\Delta^{-1})\mathrm{div}]$ be applied to the coupling term $-nE$, all the results of the present paper can be extended to the vectorial version of (1.19) which has been studied for $\omega = +\infty$ by Bidégaray [14].

2. Local-in-time Cauchy problem

The aim of this section is to show that the nonlinear system

$$\frac{1}{\omega^2}\frac{\partial^2 E}{\partial t^2} - 2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = -nE, \tag{2.1}$$

$$\frac{1}{c^2}\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2, \tag{2.2}$$

with the following initial data

$$E(0) = E_0,$$
$$\frac{\partial E}{\partial t}(0) = E_1,$$
$$n(0) = n_0,$$
$$\frac{\partial n}{\partial t}(0) = n_1,$$

is well-posed on a time interval $[0, T]$ where T is independent of ω (for the sake of clarity, we here recall that by "independent of ω", we mean quantities whose associated bound can always be bounded in turn by an ω-independent constant in the limit of large $\omega >> 1$). For this purpose, we state an appropriate mathematical setting of system (2.1) and (2.2).

2.1. Mathematical setting of the perturbed Zakharov equations

For technical convenience, we first make a change of variable on n in order to deal only with the first derivatives of E in the right-hand side of the resulting equation. In this aim, we introduce a parameter θ and compute the relation $(\mathrm{Re}((2.1) \times \bar{E}) + \theta \times (2.2))$, which expands as

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{2\omega^2}|E|^2 + \frac{\theta}{c^2}n\right] - \Delta\left[\left(\frac{1}{2} + \theta\right)|E|^2 + \theta n\right]$$
$$= \frac{1}{\omega^2}\left|\frac{\partial E}{\partial t}\right|^2 - 2\mathrm{Im}\left(\frac{\partial E}{\partial t}\bar{E}\right) - |\nabla E|^2 - n|E|^2. \tag{2.3}$$

We then choose θ as fixed by

$$\theta = \frac{c^2 - \omega^2}{2\omega^2}, \tag{2.4}$$

so that when defining $P = ((c^2 - \omega^2)/2\omega^2)n + (c^2/2\omega^2)|E|^2$ and using (2.4), Eq. (2.3) becomes

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}P - \Delta P = \frac{1}{\omega^2}\left|\frac{\partial E}{\partial t}\right|^2 - 2\mathrm{Im}\left(\frac{\partial E}{\partial t}\bar{E}\right) - |\nabla E|^2 - n|E|^2. \tag{2.5}$$

Now n is expressed in turn in terms of P and $|E|^2$ as follows:

$$n = \frac{2\omega^2}{c^2 - \omega^2}P - \frac{c^2}{c^2 - \omega^2}|E|^2 \tag{2.6}$$

and substituted into Eqs. (2.1) and (2.5) which self-consistently transform into

$$\frac{1}{\omega^2}\frac{\partial^2 E}{\partial t^2} - 2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = -\frac{2\omega^2}{c^2 - \omega^2}PE + \frac{c^2}{c^2 - \omega^2}|E|^2 E, \tag{2.7}$$

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \Delta P = \frac{1}{\omega^2}\left|\frac{\partial E}{\partial t}\right|^2 - 2\mathrm{Im}\left(\frac{\partial E}{\partial t}\bar{E}\right) - |\nabla E|^2 - \frac{2\omega^2}{c^2 - \omega^2}P|E|^2 + \frac{c^2}{c^2 - \omega^2}|E|^4. \tag{2.8}$$

The next step is to split (2.7) into the two natural "directions of propagation" attached to this wave equation. Indeed, Eq. (2.7) yields the following integral equation :

$$E(t) = S_0^{\omega}(t)E_0 + S_1^{\omega}(t)E_1 + \omega^2 \int_0^t S_1^{\omega}(t-s)\left(-\frac{2\omega^2}{c^2 - \omega^2}PE + \frac{c^2}{c^2 - \omega^2}|E|^2 E\right)(s)\,\mathrm{d}s, \tag{2.9}$$

where $S_0^{\omega}(t)$ and $S_1^{\omega}(t)$ are the semi-groups associated with the wave equation (2.7), i.e. $X(t) = S_0^{\omega}(t)E_0$ satisfies

$$\frac{1}{\omega^2}\frac{\partial^2 X}{\partial t^2} - 2\mathrm{i}\frac{\partial X}{\partial t} - \Delta X = 0, \quad X(0) = E_0, \quad \frac{\partial X}{\partial t}(0) = 0,$$

and $Y(t) = S_1^{\omega}(t)E_1$ satisfies

$$\frac{1}{\omega^2}\frac{\partial^2 Y}{\partial t^2} - 2\mathrm{i}\frac{\partial Y}{\partial t} - \Delta Y = 0, \quad Y(0) = 0, \quad \frac{\partial Y}{\partial t}(0) = E_1.$$

Here, $S_0^{\omega}(t)$ and $S_1^{\omega}(t)$ are the Fourier multipliers given by

$$\mathcal{F}(S_0^{\omega}(t))(\xi) = \frac{1 + \sqrt{1 + \xi^2/\omega^2}}{2\sqrt{1 + \xi^2/\omega^2}}\mathrm{e}^{\mathrm{i}\omega^2\left(1 - \sqrt{1+\xi^2/\omega^2}\right)t} - \frac{1 - \sqrt{1 + \xi^2/\omega^2}}{2\sqrt{1 + \xi^2/\omega^2}}\mathrm{e}^{\mathrm{i}\omega^2\left(1 + \sqrt{1+\xi^2/\omega^2}\right)t}, \tag{2.10}$$

$$\mathcal{F}(S_1^{\omega}(t))(\xi) = \frac{\mathrm{e}^{\mathrm{i}\omega^2\left(1 + \sqrt{1+\xi^2/\omega^2}\right)t} - \mathrm{e}^{\mathrm{i}\omega^2\left(1 - \sqrt{1+\xi^2/\omega^2}\right)t}}{2\mathrm{i}\omega^2\sqrt{1 + \xi^2/\omega^2}}, \tag{2.11}$$

where $\mathcal{F}$ denotes the Fourier transform with respect to the space variables. In view of (2.9)–(2.11), we therefore divide E into two components F and G having the modes of propagation $\exp(\mathrm{i}\omega^2 t(1 \pm \sqrt{1+\xi^2/\omega^2}))$, respectively, and taking place in the previously defined semi-groups, namely

$$\begin{aligned}\mathcal{F}(F(t))(\xi) = &\frac{1+\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}} \mathrm{e}^{\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)t} \mathcal{F}(E_0)(\xi) \\ &- \frac{1}{2\mathrm{i}\omega^2\sqrt{1+\xi^2/\omega^2}} \mathrm{e}^{\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)t} \mathcal{F}(E_1)(\xi) \\ &- \frac{1}{2\mathrm{i}} \int_0^t \frac{\mathrm{e}^{\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)(t-s)}}{\sqrt{1+\xi^2/\omega^2}} \mathcal{F}\left(-\frac{2\omega^2}{c^2-\omega^2}PE + \frac{c^2}{c^2-\omega^2}|E|^2E\right)(s)\,\mathrm{d}s,\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\mathcal{F}(G(t))(\xi) = &-\frac{1-\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}} \mathrm{e}^{\mathrm{i}\omega^2\left(1+\sqrt{1+\xi^2/\omega^2}\right)t} \mathcal{F}(E_0)(\xi) \\ &+ \frac{1}{2\mathrm{i}\omega^2\sqrt{1+\xi^2/\omega^2}} \mathrm{e}^{\mathrm{i}\omega^2\left(1+\sqrt{1+\xi^2/\omega^2}\right)t} \mathcal{F}(E_1)(\xi) \\ &+ \frac{1}{2\mathrm{i}} \int_0^t \frac{\mathrm{e}^{\mathrm{i}\omega^2\left(1+\sqrt{1+\xi^2/\omega^2}\right)(t-s)}}{\sqrt{1+\xi^2/\omega^2}} \mathcal{F}\left(-\frac{2\omega^2}{c^2-\omega^2}PE + \frac{c^2}{c^2-\omega^2}|E|^2E\right)(s)\,\mathrm{d}s,\end{aligned} \tag{2.13}$$

so that solution E simply reads as the amount $E = F + G$.

A straightforward calculation then shows that after taking the inverse Fourier transforms of (2.12) and (2.13), F and G respectively satisfy

$$\frac{\partial F}{\partial t} = \mathrm{i}\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)F - \frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[-\frac{2\omega^2}{c^2-\omega^2}PE + \frac{c^2}{c^2-\omega^2}|E|^2E\right], \tag{2.14}$$

and

$$\frac{\partial G}{\partial t} = \mathrm{i}\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)G + \frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[-\frac{2\omega^2}{c^2-\omega^2}PE + \frac{c^2}{c^2-\omega^2}|E|^2E\right], \tag{2.15}$$

with the initial data

$$\begin{aligned}F(0) &= \left(1+\sqrt{1-\Delta/\omega^2}\right)\left(2\sqrt{1-\Delta/\omega^2}\right)^{-1} E_0 - \frac{1}{2\mathrm{i}\omega^2}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1} E_1, \\ G(0) &= -\left(1-\sqrt{1-\Delta/\omega^2}\right)\left(2\sqrt{1-\Delta/\omega^2}\right)^{-1} E_0 + \frac{1}{2\mathrm{i}\omega^2}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1} E_1.\end{aligned} \tag{2.16}$$

At this step, the problem occurring now is that the original system (2.1) and (2.2), as well as its transformed version (2.7) and (2.8), prevents treating their solutions as belonging to the same space : if, e.g., E and n were a priori supposed to belong to H^s, it could be seen that the nonlinear contribution of the integral equation for n forces the latter solution to belong only to H^{s-1}. Because of this, the system formed by Eqs. (2.14), (2.15) and (2.8) exhibits some derivative contributions inconvenient to bound, in such a way that the standard fixed-point procedure applied to these equations cannot yield the well-posedness of the local Cauchy problem. In order to avoid this "loss of derivatives", we therefore use the Ozawa–Tsutsumi's technique expounded in [9], which consists in making some estimates working with the time-derivative of E, instead of E itself, to properly balance the space derivatives of the wave equation. To do this, we introduce the new unknown functions defined by $K \equiv \partial F/\partial t$ and $L \equiv \partial G/\partial t$, K and L being respectively solutions to

$$\begin{aligned}\frac{\partial K}{\partial t} &= \mathrm{i}\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)K \\ &-\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[-\frac{2\omega^2}{c^2-\omega^2}\left(P_t(F+G)+P(K+L)\right)\right. \\ &\left.+\frac{c^2}{c^2-\omega^2}\left(2|F+G|^2(K+L)+(F+G)^2(\bar{K}+\bar{L})\right)\right],\end{aligned} \tag{2.17}$$

$$\begin{aligned}\frac{\partial L}{\partial t} &= \mathrm{i}\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)L \\ &+\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[-\frac{2\omega^2}{c^2-\omega^2}\left(P_t(F+G)+P(K+L)\right)\right. \\ &\left.+\frac{c^2}{c^2-\omega^2}\left(2|F+G|^2(K+L)+(F+G)^2(\bar{K}+\bar{L})\right)\right].\end{aligned} \tag{2.18}$$

Using (2.14) and (2.15) we compute the initial data on K and L :

$$\begin{aligned}K(0) &= \mathrm{i}\Delta\left(2\sqrt{1-\Delta/\omega^2}\right)^{-1}E_0-\frac{1}{2}\left(1-\sqrt{1-\Delta/\omega^2}\right)\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}E_1 \\ &-\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[-\frac{2\omega^2}{c^2-\omega^2}P_0E_0+\frac{c^2}{c^2-\omega^2}|E_0|^2E_0\right],\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}L(0) &= -\mathrm{i}\Delta\left(2\sqrt{1-\Delta/\omega^2}\right)^{-1}E_0+\frac{1}{2}\left(1+\sqrt{1-\Delta/\omega^2}\right)\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}E_1 \\ &+\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[-\frac{2\omega^2}{c^2-\omega^2}P_0E_0+\frac{c^2}{c^2-\omega^2}|E_0|^2E_0\right],\end{aligned} \tag{2.20}$$

where P_0 denotes the initial datum of P, i.e.

$$P_0 = \frac{c^2-\omega^2}{2\omega^2}n_0+\frac{c^2}{2\omega^2}|E_0|^2, \tag{2.21}$$

and where

$$P_1 = \frac{c^2-\omega^2}{2\omega^2}n_1+\frac{c^2}{\omega^2}\mathrm{Re}(\bar{E}_0E_1) \tag{2.22}$$

denotes the initial datum of $\partial P/\partial t$. We also substitute E by $(F+G)$ and $\partial E/\partial t$ by $(K+L)$ into Eq. (2.8), which gives, keeping the term $-|\nabla E|^2$ untouched

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}-\Delta P = \frac{1}{\omega^2}|K+L|^2-2\mathrm{Im}\left((K+L)(\bar{F}+\bar{G})\right)-|\nabla E|^2$$

$$-\frac{2\omega^2}{c^2-\omega^2}P|F+G|^2+\frac{c^2}{c^2-\omega^2}|F+G|^4. \tag{2.23}$$

In Eqs. (2.17), (2.18) and (2.23), we then replace F and G by $F = F(0) + \int_0^t K(s)\,\mathrm{d}s$ and $G = G(0) + \int_0^t L(s)\,\mathrm{d}s$ respectively, while $-|\nabla E|^2$ in (2.23) is expanded writing $\nabla E = \nabla F + \nabla G$. Analogously to Ref. [9], we finally add $-\mathrm{i}F$ to both sides of (2.14) in order to avoid the low-frequency divergence of the operator $\left(1-\sqrt{1-\Delta/\omega^2}\right)^{-1}$, and after solving for F, we obtain

$$\begin{aligned}
F = &\left(\mathrm{i}\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)-\mathrm{i}\right)^{-1}\left\{K-\mathrm{i}\left(F(0)+\int_0^t K(s)\,\mathrm{d}s\right)\right.\\
&+\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[\frac{-2\omega^2}{c^2-\omega^2}P\left(F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right)\right.\\
&+\frac{c^2}{c^2-\omega^2}\left|F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right|^2\\
&\left.\left.\times\left(F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right)\right]\right\}\\
\equiv\; &\mathcal{T}_1^{\omega}(K,L,P).
\end{aligned} \tag{2.24}$$

Similarly, using Eq. (2.15), one gets

$$\begin{aligned}
G = &\left(\mathrm{i}\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)\right)^{-1}\\
&\times\left\{L-\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[\frac{-2\omega^2}{c^2-\omega^2}P\left(F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right)\right.\right.\\
&+\frac{c^2}{c^2-\omega^2}\left|F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right|^2\\
&\left.\left.\times\left(F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right)\right]\right\}\\
\equiv\; &\mathcal{T}_2^{\omega}(K,L,P).
\end{aligned} \tag{2.25}$$

The appropriate system, on which the forthcoming fixed-point method will be applied, therefore reads

$$\frac{\partial K}{\partial t} = \mathrm{i}\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)K + M_1(K,L,P,P_t), \tag{2.26}$$

$$\frac{\partial L}{\partial t} = \mathrm{i}\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)L - M_1(K,L,P,P_t), \tag{2.27}$$

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \Delta P = M_2(K,L,P), \tag{2.28}$$

where

$$M_1(K,L,P,P_t)=-\frac{1}{2\mathrm{i}}\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}\left[\frac{-2\omega^2}{c^2-\omega^2}\left(P_t\left(F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right)\right.\right.$$
$$\left.+P(K+L)\right)$$
$$+\frac{c^2}{c^2-\omega^2}\left(2|F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s|^2(K+L)\right.$$
$$\left.\left.+(F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s)^2(\bar{K}+\bar{L})\right)\right] \tag{2.29}$$

and

$$M_2(K,L,P)=\frac{1}{\omega^2}|L+K|^2-2\mathrm{Im}\left((L+K)(\bar{F}(0)+\bar{G}(0)+\int_0^t (\bar{K}(s)+\bar{L}(s))\,\mathrm{d}s)\right)$$
$$-\left|\nabla T_1^{\omega}(K,L,P)+\nabla T_2^{\omega}(K,L,P)\right|^2$$
$$-\frac{2\omega^2}{c^2-\omega^2}P\left|F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right|^2$$
$$+\frac{c^2}{c^2-\omega^2}\left|F(0)+G(0)+\int_0^t (K(s)+L(s))\,\mathrm{d}s\right|^4. \tag{2.30}$$

Following the procedure developed in Ozawa–Tsutsumi [9], it can finally be shown that as soon as all the terms in (2.29) and (2.30) make sense, Eqs. (2.26)–(2.28) are equivalent to (2.1) and (2.2).

2.2. *Existence and uniqueness*

In what follows, X denotes the product space $X = H^s \times H^s \times H^{s+1} \times H^s$ with $s > d/2$ and U is a generic element of X such as $U = (K, L, P, P_t)$. The norm in X is henceforth defined by

$$|U|_X = |K|_{H^s} + |L|_{H^s} + |P|_{H^{s+1}} + |P_t|_{H^s},$$

and the forthcoming quantity C will refer to a positive constant which can change from one line to another one in the remaining part of the analysis.

The aim of this section is to prove the following results.

Theorem 1. Let $U_0 \equiv (K_0, L_0, P_0, P_1) \in X$, then there exists $T > 0$, depending only on $|U_0|_X$, such that there exists a unique solution $U = (K, L, P, P_t)$ to (2.26)–(2.28) satisfying $U \in C^0([0, T]; X)$ and $U(0) = U_0$. Moreover U depends continuously on U_0 and $|U|_{L^\infty(0,T;X)}$ is bounded independently of ω.

Corollary 1. Let $E_0 \in H^{s+2}$, $E_1 \in H^s$, $n_0 \in H^{s+1}$, $n_1 \in H^s$ and suppose that $|E_1|_{H^{s+1}}/\omega$ is uniformly bounded, then there exists $T > 0$, depending only on $|E_0|_{H^{s+2}} + |E_1|_{H^s} + \frac{1}{\omega}|E_1|_{H^{s+1}} + |n_0|_{H^{s+1}} + |n_1|_{H^s}$, such that there exists a unique solution to (2.1) and (2.2) satisfying $E \in C^0([0,T]; H^{s+2}) \bigcap C^1([0,T]; H^s)$ and $n \in C^0([0,T]; H^{s+1}) \bigcap C^1([0,T]; H^s)$ with $E(0) = E_0$, $\frac{\partial E}{\partial t}(0) = E_1$, $n(0) = n_0$ and $\frac{\partial n}{\partial t}(0) = n_1$.
Moreover E and n are bounded in their respective spaces independently of ω.

Proof of Theorem 1. Let us first introduce the following functionals

$$N_1[K,L,P,P_t] \equiv e^{i\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)t} K_0 + \int_0^t e^{i\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)(t-s)} M_1(K,L,P,P_t)(s)\,ds, \tag{2.31}$$

$$N_2[K,L,P,P_t] \equiv e^{i\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)t} L_0 - \int_0^t e^{i\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)(t-s)} M_1(K,L,P,P_t)(s)\,ds, \tag{2.32}$$

$$N_3[K,L,P,P_t] \equiv \cos(c(-\Delta)^{1/2}t)P_0 + \frac{\sin(c(-\Delta)^{1/2}t)}{c}(-\Delta)^{-1/2}P_1 + c\int_0^t \sin(c(-\Delta)^{1/2}(t-s))(-\Delta)^{-1/2}M_2(K,L,P)(s)\,ds, \tag{2.33}$$

$$N_4[K,L,P,P_t] \equiv -c(-\Delta)^{1/2}\sin(c(-\Delta)^{1/2}t)P_0 + \cos(c(-\Delta)^{1/2}t)P_1 + c^2\int_0^t \cos(c(-\Delta)^{1/2}(t-s))M_2(K,L,P)(s)\,ds, \tag{2.34}$$

so that system (2.26)–(2.28) remains equivalent to $N[K,L,P,P_t] = (K,L,P,P_t)$ with $N = (N_1,N_2,N_3,N_4)$. Using the fact that M_1 and M_2 are continuous on X, we can easily perform a standard fixed-point procedure as in [9], in order to show that if T is small enough, equation $N[K,L,P,P_t] = (K,L,P,P_t)$ possesses a unique solution in $L^\infty(0,T;X)$, which is the result of Theorem 1. □

Proof of Corollary 1. First we remark that if $E_0 \in H^{s+2}$, $E_1 \in H^s$, $n_0 \in H^{s+1}$ and $n_1 \in H^s$, then (2.19)–(2.22) imply that $(K(0), L(0), P_0, P_1) \in X$. Hence, the result of Theorem 1 applies.
Afterwards, it turns out from Eqs. (2.24) and (2.25) that F and $G \in L^\infty(0,T;H^{s+1})$. Moreover, since $P \in L^\infty(0,T;H^{s+1})$, it can be seen from the expression (2.6) that n belongs in turn to $L^\infty(0,T;H^{s+1})$ whereas $\partial P/\partial t \in L^\infty(0,T;H^s)$.
On the other hand, we notice that by (2.29), $(1/\omega)M_1(K,L,P,P_t)$ lies in $L^\infty(0,T;H^{s+1})$ since the operator $(1/\omega)\left(\sqrt{1-\Delta/\omega^2}\right)^{-1}$ is a smoothing operator of order 1 (independent of ω). In addition, since $|E_1|_{H^{s+1}}/\omega$ is

bounded, we deduce from (2.19) and (2.20), that $(1/\omega)K$ and $(1/\omega)L$ are bounded in turn in $L^{\infty}(0,T;H^{s+1})$. In view of (2.24) and (2.25), we finally introduce

$$\tilde{F} = \left(\mathrm{i}\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)-\mathrm{i}\right)F,$$

and

$$\tilde{G} = \left(\mathrm{i}\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)\right)G,$$

so that $\tilde{F}$ and $\tilde{G}$ are bounded in $L^{\infty}(0,T;H^{s})$, and therefore, $\tilde{F}/\omega$ together with $\tilde{G}/\omega$ are bounded in $L^{\infty}(0,T;H^{s+1})$ independently of ω.

Lemma 1. F and G are bounded in $L^{\infty}(0,T;H^{s+2})$ independently of ω.

Proof. Since the function $|\xi|/\mathrm{i}\omega\left(1+\sqrt{1+\xi^2/\omega^2}\right)$ is bounded independently of ω, it is clear that the previous estimate $\left|\tilde{G}/\omega\right|_{L^{\infty}(0,T;H^{s+1})} \leq C$ immediately leads to the result for G. This property, however, does not apply to F, since the function $|\xi|\omega/(\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)-1)$ is not bounded independently of ω. To find an accurate bound for F, we have to estimate the integral $\int_{\xi\in\mathbf{R}^d}|\xi|^{2s+4}|\mathcal{F}(F)|^2(\xi)\,\mathrm{d}\xi$ by dividing the whole range of integration into the two domains $\{|\xi|<\omega\}$ and $\{|\xi|>\omega\}$ and set

$$I_1 = \int\limits_{|\xi|<\omega} |\xi|^{2s+4}\left|\frac{1}{\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)-\mathrm{i}}\right|^2 |\mathcal{F}(\tilde{F})|^2\,\mathrm{d}\xi,$$

and

$$I_2 = \int\limits_{|\xi|>\omega} |\xi|^{2s+4}\left|\frac{1}{\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)-\mathrm{i}}\right|^2 |\mathcal{F}(\tilde{F})|^2\,\mathrm{d}\xi.$$

In the range $\{|\xi|<\omega\}$, we observe that the function $|\xi| \mapsto \xi^2/|\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)-\mathrm{i}|$ is increasing, so that one has

$$I_1 \leq C\int\limits_{|\xi|<\omega}|\xi|^{2s}|\mathcal{F}(\tilde{F})|^2\,\mathrm{d}\xi \leq C|\tilde{F}|_{L^{\infty}(0,T;H^s)}, \tag{2.35}$$

while in the complementary range $\{|\xi|>\omega\}$, the function
$|\xi| \mapsto |\xi|/|\mathrm{i}\omega^2\left(1-\sqrt{1+\xi^2/\omega^2}\right)-\mathrm{i}|$ is decreasing, from which the estimate

$$I_2 \leq C\int\limits_{|\xi|>\omega}|\xi|^{2s+2}\left|\mathcal{F}\left(\frac{\tilde{F}}{\omega}\right)\right|^2\mathrm{d}\xi \leq C\left|\frac{\tilde{F}}{\omega}\right|_{L^{\infty}(0,T;H^{s+1})} \tag{2.36}$$

follows. The inequalities (2.35) and (2.36) finally achieve the proof of Lemma 1.

From the above results, one thus concludes that E remains bounded independently of ω. Let us finally remark that the same property applies to n, as P and P_t both satisfy the same crietrium of boundedness and since n expresses in terms of E and P through the relation (2.6), which achieves the proof of Corollary 1. □

Remark 1. The maximal existence time of the solution T_s^ω seems to depend on the regularity of the initial data. In fact, if $U_0 \in X^s \cap X^{s'}$ (with obvious notations), it can be checked that $T_s^\omega = T_{s'}^\omega$, as is usually the case when the solution is issued from a fixed-point method.

Remark 2. For the well-posedness of the Cauchy problem, it was not strictly speaking necessary to split E into F and G. However, it was worth introducing such a splitting because as it will be seen in Section 3, the latter decomposition will be of utmost important technical interest when detailing the converging and the oscillatory parts of $\partial E/\partial t$ in the limit process $\omega^2 \to \infty$.

3. The route towards the time-envelope approximation

The aim of this section is to investigate the limit $\omega^2 \to \infty$ in

$$\frac{1}{\omega^2}\frac{\partial^2 E^\omega}{\partial t^2} - 2\mathrm{i}\frac{\partial E^\omega}{\partial t} - \Delta E^\omega = -n^\omega E^\omega, \tag{3.1}$$

$$\frac{1}{c^2}\frac{\partial^2 n^\omega}{\partial t^2} - \Delta n^\omega = \Delta|E^\omega|^2, \tag{3.2}$$

or in the framework of the equivalent formulation

$$\frac{\partial K^\omega}{\partial t} = \mathrm{i}\omega^2\left(1 - \sqrt{1-\Delta/\omega^2}\right)K^\omega + M_1(K^\omega, L^\omega, P^\omega, P_t^\omega), \tag{3.3}$$

$$\frac{\partial L^\omega}{\partial t} = \mathrm{i}\omega^2\left(1 + \sqrt{1-\Delta/\omega^2}\right)L^\omega - M_1(K^\omega, L^\omega, P^\omega, P_t^\omega), \tag{3.4}$$

$$\frac{1}{c^2}\frac{\partial^2 P^\omega}{\partial t^2} - \Delta P^\omega = M_2(K^\omega, L^\omega, P^\omega). \tag{3.5}$$

The main result can be expressed as the following where arrows henceforth refer to the limit $\omega^2 \to \infty$, and where, for the sake of convenience, the functions K, P, P_t and E now denote their respective limit values.

Theorem 2. (i) Let $K_0, L_0, P_1 \in H^s$ and $P_0 \in H^{s+1}$ together with $(1/\omega)|E_1|_{H^{s+1}} \to 0$ as $\omega \to \infty$, then there exists $T > 0$ such that the following convergence results hold in $L^\infty(0, T; H^s)$:

$$K^\omega \to K,$$

$$L^\omega - \mathrm{e}^{2\mathrm{i}\omega^2 t} w(x,t) \to 0,$$

$$P_t^\omega \to P_t,$$

and

$$P^\omega \to P$$

in $L^\infty(0, T; H^{s+1})$ with $K = \partial E/\partial t$, where E and P satisfy

$$-2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = 2PE, \tag{3.6}$$

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \Delta P = -\frac{1}{2}\Delta|E|^2, \tag{3.7}$$

and w is the solution to

$$2\mathrm{i}\frac{\partial w}{\partial t} - \Delta w = 2Pw, \qquad w(x,0) = L_0. \tag{3.8}$$

(ii) Moreover, if T^ω is the existence time associated to (3.1) and (3.2) and T^∞ the existence time associated to (3.6) and (3.7) then

$$\liminf_{\omega\to\infty} T^\omega \geq T^\infty,$$

and for all $T_1 < T^\infty$ the above convergences hold on $[0, T_1]$.

(iii) If $\liminf_{\omega\to\infty} T^\omega > T^\infty$, then

$$\lim_{\omega\to\infty}(|K^\omega|_{H^s} + |L^\omega|_{H^s} + |P^\omega|_{H^{s+1}} + |P_t^\omega|_{H^s})(T^\infty) = +\infty.$$

Proof (sketch). Basically the proof follows the scheme developed in [4] and is based upon the fixed-point equations introduced in Section 2.2. We first prove a local version of the convergence result (i). Points (ii) and (iii) can be inferred from repeating the same arguments as in [4] and using the fact that for ω large enough, K^ω, L^ω, P^ω and P_t^ω are controlled by the solution of the limit system (3.6) and (3.7). Results similar to (ii) have been proved by Constantin [11] in the context of the Euler and Navier–Stokes equations and adapted to the Zakharov equations in [10]. Furthermore, the theorem remains valid if (K_0, L_0, P_0, P_1) depend on ω and converge in $H^s \times H^s \times H^{s+1} \times H^s$ as ω tends to ∞.

Proof of (i). Let K and P be respectively solution to

$$K = \mathrm{e}^{(\mathrm{i}/2)\Delta t}K_0 - \int_0^t \mathrm{e}^{(\mathrm{i}/2)\Delta(t-s)}\left\{\frac{1}{2\mathrm{i}}(2P_tE + 2PK)(s)\right\}\mathrm{d}s, \tag{3.9}$$

and

$$P = \cos(c(-\Delta)^{1/2}t)P_0 + \frac{1}{c}\sin(c(-\Delta)^{1/2}t)(-\Delta)^{-1/2}P_1 + c\int_0^t \sin(c(-\Delta)^{1/2}(t-s))(-\Delta)^{-1/2}(-2\,\mathrm{Im}(K\bar{E}) - |\nabla E|^2 + 2PE)(s)\,\mathrm{d}s, \tag{3.10}$$

with $K = \partial E/\partial t$. The system formed by Eqs. (3.9) and (3.10) may be seen to remain equivalent to the unperturbed Zakharov equations (3.6) and (3.7). For what concerns the perturbed counterpart of the latter, we here recall that K^ω, L^ω, P^ω and P_t^ω verify the following set of integral equations :

$$K^{\omega} = e^{i\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)t} K_0 + \int_0^t e^{i\omega^2\left(1-\sqrt{1-\Delta/\omega^2}\right)(t-s)} M_1(K^{\omega}, L^{\omega}, P^{\omega}, P_t^{\omega})(s)\,ds, \tag{3.11}$$

$$L^{\omega} = e^{i\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)t} L_0 - \int_0^t e^{i\omega^2\left(1+\sqrt{1-\Delta/\omega^2}\right)(t-s)} M_1(K^{\omega}, L^{\omega}, P^{\omega}, P_t^{\omega})(s)\,ds, \tag{3.12}$$

$$\begin{aligned} P^{\omega} &= \cos(c(-\Delta)^{1/2}t)P_0 + \frac{1}{c}\sin(c(-\Delta)^{1/2}t)(-\Delta)^{-1/2}P_1 \\ &\quad + c\int_0^t \sin(c(-\Delta)^{1/2}(t-s))(-\Delta)^{-1/2}M_2(K^{\omega}, L^{\omega}, P^{\omega})(s)\,ds, \end{aligned} \tag{3.13}$$

$$\begin{aligned} P_t^{\omega} &= -c(-\Delta)^{1/2}\sin(c(-\Delta)^{1/2}t)P_0 + \cos(c(-\Delta)^{1/2}t)P_1 \\ &\quad + c^2\int_0^t \cos(c(-\Delta)^{1/2}(t-s))M_2(K^{\omega}, L^{\omega}, P^{\omega})(s)\,ds. \end{aligned} \tag{3.14}$$

We moreover introduce the following set of unknowns :

$$K_1^{\omega} = K^{\omega} - K,$$

$$L_1^{\omega} = L^{\omega} - e^{2i\omega^2 t} w(x,t),$$

$$P_1^{\omega} = P^{\omega} - P,$$

$$P_{1t}^{\omega} = P_t^{\omega} - P_t,$$

where w is defined by (3.8). Subtracting then (3.9) from (3.11), (3.10) from (3.13), and $\frac{\partial}{\partial t}$(3.10) from (3.14) thus provides the integral equations for K_1^{ω}, P_1^{ω} and P_{1t}^{ω}. Besides, subtracting $e^{2i\omega^2 t}w(x,t)$ from each side of (3.12), we obtain an integral equation for L_1^{ω}. With this set of equations, it is easy, although technical, to show that if T is small enough, one finds

$$|K_1^{\omega}|_{L^{\infty}(0,T;H^s)} + |L_1^{\omega}|_{L^{\infty}(0,T;H^s)} + |P_1^{\omega}|_{L^{\infty}(0,T;H^{s+1})} + |P_{1t}^{\omega}|_{L^{\infty}(0,T;H^s)} \le h(\omega), \tag{3.15}$$

where $h(\omega)$ tends to zero as ω tends to $+\infty$. One of the key methods used to obtain (3.15) is, similarly to the analysis performed in [4], the convergence result:

$$\int_0^t e^{2i\omega^2\sigma} j_{\omega}(x,\sigma)\,d\sigma \to 0 \text{ in } L^{\infty}(0,T;H^s) \tag{3.16}$$

for any sequence $j_{\omega} \to j$ in $L^{\infty}(0,T;H^s)$ as $\omega \to \infty$, which follows from a classical nonstationary phase lemma, and ends the sketch of the proof of Theorem 2. □

We now briefly explain how to find the corrector for L^{ω} : let us first define $S(t)$ as being the unitary group associated to $-2i\partial_t - \Delta$. Since the contribution $(e^{i\omega^2(1+\sqrt{1-\Delta/\omega^2})t}L_0 - e^{2i\omega^2 t}S(-t)L_0)$ tends to zero in $L^{\infty}(0,T;H^s)$ as

$\omega^2 \to \infty$, one easily deduces that L^ω is an oscillatory function which has therefore to be sought under the form given by Theorem 2. Now, thanks to (3.12) and (2.29), w has to formally satisfy in the limit $\omega^2 \to \infty$

$$e^{2i\omega^2 t} w(x,t) \approx e^{2i\omega^2 t} S(-t) L_0 + \int_0^t e^{2i\omega^2 (t-s)} S(-t+s) \frac{i}{i} P e^{2i\omega^2 s} w(x,s)\, ds, \tag{3.17}$$

where the property (3.16) has intensively been used. This leads to

$$w(x,t) = S(-t) L_0 + \int_0^t S(-t+s) \frac{1}{i} P w(x,s)\, ds, \tag{3.18}$$

which is equivalent to (3.8).

Remark 3. Note that L^ω tends to zero if and only if $L_0 \to 0$, i.e. as concluded from (2.20), when the limit values of the initial data satisfy the compatibility condition imposed by the unperturbed Zakharov equation (3.6)

$$-i\Delta E_0 + 2E_1 - 2iP_0 E_0 = 0.$$

From this property, it can easily be guessed that as long as the compatibility condition is not fulfilled, $\partial E^\omega/\partial t$ will not converge towards its time-enveloped limit $\partial E/\partial t$. To this point, we emphasize that an analogous property characterizes the behavior of the perturbed solution to the nonlinear Schrödinger equation corresponding to the formal limit $c^2 \to +\infty$ in (2.2), as studied in [4].

Concerning the original system (3.1) and (3.2) the same method as for Corollary 1 enables us to deduce from Theorem 2 the following corollary:

Corollary 2. Let $E_0 \in H^{s+2}$, $E_1 \in H^s$, $n_0 \in H^{s+1}$, $n_1 \in H^s$ and suppose that $(|E_1|_{H^{s+1}}/\omega) \to 0$ as $\omega \to \infty$, then there exists $T > 0$ such that the following convergence results hold

$$E^\omega \to E \text{ in } L^\infty(0,T;H^{s+2}),$$

$$E_t^\omega - E_t - e^{2i\omega^2 t} w(x,t) \to 0 \text{ in } L^\infty(0,T;H^s),$$

$$n^\omega \to n \text{ in } L^\infty(0,T;H^{s+1}),$$

$$n_t^\omega \to n_t \text{ in } L^\infty(0,T;H^s),$$

where E and n are solutions to

$$-2i\frac{\partial E}{\partial t} - \Delta E = -nE, \tag{3.19}$$

$$\frac{1}{c^2}\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2. \tag{3.20}$$

4. About blow-up and global existence

It is well known that in spite of several attempts at indicating the possible development of a singularity, no definite conclusion on the existence of a finite time blow-up for the unperturbed Zakharov equations (3.19) and (3.20) has been sorted out so far. By means of a bifurcation method, Glangetas and Merle [12], however, proved in the two-dimensional case the existence of initial data E_0, n_0, n_1 such that the corresponding solution to (3.19) and (3.20) blows up in a finite time with a self-similar shape. In addition, Merle [13] demonstrated the occurrence of an infinite time blow up in the three-dimensional case, by using an adapted Virial identity. Let us recall in this respect that, for every space dimension number, $|E|_{L^2}(t)$ is constant, and furthermore, in the one-dimensional case, the solution to (3.19) and (3.20) is global and uniformly bounded in time [9]. Therefore, in contrast to the usual Zakharov equations, it is worth mentioning that for what concerns the perturbed equation set (3.1) and (3.2), we can really prove that for some initial data, the L^2 norm of solution E^ω blows up at least in an infinite time, whatever the space dimension number may be. To prove this result, we preliminarily emphasize that (3.1) and (3.2) consist of a Hamiltonian system admitting the following two invariants, namely the charge

$$Q_\omega \equiv \int_{\mathbf{R}^d} |E^\omega|^2 - \frac{1}{\omega^2}\mathrm{Im}\int_{\mathbf{R}^d} \frac{\partial E^\omega}{\partial t}\bar{E}^\omega, \tag{4.1}$$

and the energy (Hamiltonian) integral

$$\mathcal{E}_\omega \equiv \int_{\mathbf{R}^d} \left(\frac{1}{\omega^2}\left|\frac{\partial E^\omega}{\partial t}\right|^2 + |\nabla E^\omega|^2 + n^\omega|E^\omega|^2 + \frac{c^2}{2}(\nabla\psi^\omega)^2 + \tfrac{1}{2}(n^\omega)^2\right). \tag{4.2}$$

In (4.2), ψ^ω denotes the real potential function associated with the ion-sound flow and restitutes the second Zakharov equation by combining the continuity equations

$$c^2\Delta\psi^\omega = n_t^\omega$$

and

$$\psi_t^\omega = n^\omega + |E^\omega|^2.$$

It can be noted that those relations simply follow from deriving the Hamilton–Lagrange equation for n^ω supplemented by the continuity relation on ψ^ω. The blow-up result can then be stated as follows :

Theorem 3. Let us ensure that the solutions defined under the starting hypothesis of Corollary 2 exist for every time. Suppose moreover that $n_1 \in \dot{H}^{-1}(\mathbf{R}^d)$ and introduce the quantity $I(t) = \frac{1}{2}\int_{\mathbf{R}^d}|E^\omega|^2$. If the conserved quantities (4.1) and (4.2) satisfy one of the two following conditions :
(C1) $\mathcal{E}_\omega + 2\omega^2 Q_\omega < 0$
(C2) $\mathcal{E}_\omega + 2\omega^2 Q_\omega = 0, \quad \dot{I}(0) > 0,$
then

$$\lim_{t\to\infty} |E^\omega|_{L^2} = +\infty.$$

Proof. We start with the following transform

$$E^\omega = \chi^\omega \exp(\mathrm{i}\omega^2 t) \tag{4.3}$$

that we insert into Eqs. (3.1) and (3.2) to obtain the new system

$$\frac{1}{\omega^2}\frac{\partial^2\chi^\omega}{\partial t^2}+\omega^2\chi^\omega-\Delta\chi^\omega=-n^\omega\chi^\omega, \tag{4.4}$$

$$\frac{1}{c^2}\frac{\partial^2 n^\omega}{\partial t^2}-\Delta n^\omega=\Delta|\chi^\omega|^2. \tag{4.5}$$

These equations admit in turn the following energy integral

$$\mathcal{E}'_\omega \equiv \int\limits_{\mathbf{R}^d}\left(\frac{1}{\omega^2}\left|\frac{\partial\chi^\omega}{\partial t}\right|^2+\omega^2|\chi^\omega|^2+|\nabla\chi^\omega|^2+n^\omega|\chi^\omega|^2+\frac{c^2}{2}(\nabla\psi^\omega)^2+\tfrac{1}{2}(n^\omega)^2\right), \tag{4.6}$$

where the function ψ^ω expresses as before by simply replacing the densities $|E^\omega|^2 \to |\chi^\omega|^2$. Following the concavity method introduced by Levine [15], we now multiply (4.4) by $\bar{\chi}^\omega$ and integrate in space the real part of the resulting equation to get

$$\frac{1}{2\omega^2}\frac{\partial^2}{\partial t^2}\int\limits_{\mathbf{R}^d}|\chi^\omega|^2=\frac{2}{\omega^2}\int\limits_{\mathbf{R}^d}\left|\frac{\partial\chi^\omega}{\partial t}\right|^2-\mathcal{E}'_\omega+\frac{1}{2}\int\limits_{\mathbf{R}^d}(n^\omega)^2+\frac{c^2}{2}\int\limits_{\mathbf{R}^d}(\nabla\psi^\omega)^2. \tag{4.7}$$

Next, introducing the definition $I(t)=\frac{1}{2}\int_{\mathbf{R}^d}|E^\omega|^2=\frac{1}{2}\int_{\mathbf{R}^d}|\chi^\omega|^2$, Eq. (4.7) is easily checked to reduce to

$$I\ddot{I}\geq(\dot{I})^2-\omega^2\mathcal{E}'_\omega I \tag{4.8}$$

after using the Cauchy–Schwarz inequality $\int_{\mathbf{R}^d}|\chi_t^\omega|^2\int_{\mathbf{R}^d}|\chi^\omega|^2\geq(\dot{I})^2$. Keeping in mind that $\dot{I}(t)=\int_{\mathbf{R}^d}\mathrm{Re}(\bar{\chi}^\omega\chi_t^\omega)$ $\int_{\mathbf{R}^d}\mathrm{Re}(\bar{E}^\omega E_t^\omega)$, we now consider some negative-energy states characterized by the inequality $\mathcal{E}'_\omega\leq 0$ and investigate the two following cases :

(a) *Suppose* $\dot{I}(0)>0$ *and* $\mathcal{E}'_\omega\leq 0$: then estimate (4.8) leads to the result since $I(t)$ diverges faster than e^{Ct} $(C>0)$ with

$$I(t)\geq I(0)\exp\left(\frac{\dot{I}(0)}{I(0)}t\right). \tag{4.9}$$

(b) *Suppose* $\dot{I}(0)\leq 0$ *and* $\mathcal{E}'_\omega<0$: then we introduce the positive *ansatz*

$$H(t)\equiv I(t)-(\mathcal{E}'_\omega\omega^2/2)(t+\tau)^2$$

(τ to be discussed briefly on) to get the following inequality using (4.8)

$$H\ddot{H}-(\dot{H})^2\geq-(\omega^2\mathcal{E}'_\omega/2I)[(t+\tau)\dot{I}-2I]^2\geq 0.$$

Integrating the latter twice in time, we obtain

$$H(t)\geq H(0)\exp\left(\frac{\dot{H}(0)}{H(0)}t\right)$$

that assures $I(t)$ to diverge faster than e^{Ct} as $t\to+\infty$ whenever the parameter τ is chosen to ensure $\dot{H}(0)>0$, i.e. for $\tau>\dot{I}(0)/(\omega^2\mathcal{E}'_\omega)$. To achieve this proof, we notice that the above steps (a) and (b) cover both situations (C1) and (C2) stated in the theorem whose underlying hypothesis $\mathcal{E}_\omega+2\omega^2Q_\omega\leq 0$ clearly arises when one explicitly expands the basic constraint $\mathcal{E}'_\omega\leq 0$ in terms of the primary solution E^ω related to χ^ω by (4.3). □

Remark 4. This result applies for every space dimension and thus predicts that solutions satisfying initially the assumptions (C1) and (C2) become singular as $t \to +\infty$. The blow-up described here does not certainly resemble the one governed by the unperturbed Zakharov equations (3.19) and (3.20) which satisfy $|E|_{L^2}(t) = |E_0|_{L^2}$, while in our case $|E^\omega(t)|_{L^2} \to +\infty$ as $t \to +\infty$. Unlike the finite-time blow-up revealed in [4], the blow-up time is here infinite, because, by comparison with the former analysis, the bound from below found out in the inequality (4.9) does not naturally diverge at a finite instant. Even though Theorem 3 does not give an explicit finite time blow-up result, one can nevertheless expect the explosion of the L^2 norm of the solutions in a way similar to the one discovered in [4] in the framework of the nonlinear Schrödinger equation, i.e. with a time-increasing L^2 norm. Furthermore, one can refind another interesting property already mentioned in [4] : this concerns the important feature according to which solutions to Eqs. (3.1) and (3.2) may blow up *provided that* ω^2 *should not be too large*. In the opposite case, i.e. in the limit $\omega^2 \to +\infty$, one easily sees that the amount $\mathcal{E}_\omega + 2\omega^2 Q_\omega$ should then become close to $2\omega^2 I(t)$, which would make the assumptions (C1) and (C2) quite invalid. Reversely, from the latter hypotheses, the previous theorem can be concluded to only apply to some values of ω lying in the range

$$\omega \le \sqrt{\frac{-\mathcal{E}_\omega}{2Q_\omega}}.$$

In connection with this result, we restore the global existence of smooth solutions to the usual Zakharov equations expressed in space dimension 1 by means of the following proposition.

Proposition 1. Let $d = 1$, E_0, E_1, n_0 and n_1 as in Corollary 2 ($s > \frac{1}{2}$). We moreover suppose that $n_1 \in \dot{H}^{-1}(\mathbb{R})$. If ω is large enough, the solution to (3.1) and (3.2) is global in time.

Proof. Unlike the case $\omega = +\infty$, the L^2 norm of E^ω is not conserved. In view of the above blow-up result, the first point is here to prove that for ω sufficiently large, $|E^\omega|_{L^2}$ is bounded independently of the time t. For this purpose, we introduce the quantity $\mathcal{H}$ defined by

$$\mathcal{H} \equiv Q_\omega + \frac{1}{2\omega^2}\mathcal{E}_\omega,$$

which is constant in time.

Estimating

$$\frac{1}{\omega^2}\left|\frac{\partial E^\omega}{\partial t}E^\omega\right| \le \frac{1}{2\omega^4}\left|\frac{\partial E^\omega}{\partial t}\right|^2 + \frac{1}{2}|E^\omega|^2 \quad \text{and} \quad n^\omega|E^\omega|^2 \le \frac{1}{2}(n^\omega)^2 + \frac{1}{2}|E^\omega|^4,$$

we get

$$\mathcal{H} \ge \int \frac{1}{2}|E^\omega|^2 + \frac{1}{2\omega^2}\left(|\nabla E^\omega|^2 - \frac{1}{2}|E^\omega|^4\right),$$

where, for the sake of clarity, the integral symbol with no other specification henceforth refers to an integration over the whole 1-D space. We make use of the Gagliardo–Nirenberg inequality to sort out

$$\mathcal{H} \ge \int \frac{1}{2}|E^\omega|^2 - \frac{C}{\omega^2}\left(\int |E^\omega|^2\right)^3.$$

From this last inequality, and using the continuity of the map $t \mapsto |E^\omega|_{L^2}$, it is easy to deduce that if ω is large enough, then the L^2 norm of E^ω is bounded. Employing this inequality, the same type of energy estimates as the ones performed in Sulem and Sulem [8] yields the global existence result. Indeed, let us now rewrite the system as

$$\frac{1}{\omega^2}E^\omega_{tt} - 2\mathrm{i}E^\omega_t - E^\omega_{xx} = -n^\omega E^\omega, \tag{4.10}$$

$$n^\omega_t = -cV^\omega_x, \tag{4.11}$$

$$V^\omega_t = -c(n^\omega + |E^\omega|^2)_x, \tag{4.12}$$

where $V^\omega = -c\psi^\omega_x$. We know that $|E^\omega|_{L^2}$ is bounded and by virtue of (4.2), as we deal with the subcritical 1-D case, $(1/\omega)E^\omega_t$, E^ω_x, V^ω and n^ω are bounded in L^2, uniformly with respect to t and ω.

We first compute $\partial_t(4.11) \times n^\omega_t + \partial_t(4.12) \times V^\omega_t$ and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int |n^\omega_t|^2 + |V^\omega_t|^2\right) + 2\int |E^\omega|^2_t n^\omega_{tt} = 0. \tag{4.13}$$

Next, integrating $\mathrm{Im}(\partial_t(4.10) \times \bar{E}^\omega_t)$ over space leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |E^\omega_t|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\frac{1}{\omega^2}\int E^\omega_{tt}\bar{E}^\omega_t = \mathrm{Im}\int n^\omega_t E^\omega \bar{E}^\omega_t. \tag{4.14}$$

Finally, computing $\mathrm{Re}(\partial_t(4.10) \times \bar{E}^\omega_{tt})$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\omega^2}\int |E^\omega_{tt}|^2 + \int |E^\omega_{xt}|^2\right) = -\int n^\omega_t(|E^\omega|^2_{tt} - 2|E^\omega_t|^2) - \int n^\omega(|E^\omega_t|^2_t). \tag{4.15}$$

Integrating the linear combination (4.13)+(4.14)+2×(4.15) over time yields

$$\begin{aligned}
&|n^\omega_t|^2_{L^2} + |V^\omega_t|^2_{L^2} + |E^\omega_t|^2_{L^2} - \mathrm{Im}\left(\frac{1}{\omega^2}\int E^\omega_{tt}\bar{E}^\omega_t\right) \\
&+\frac{2}{\omega^2}|E^\omega_{tt}|^2_{L^2} + 2|E^\omega_{xt}|^2_{L^2} + 2\int (n^\omega_t(|E^\omega|^2)_t + n^\omega|E^\omega_t|^2) \\
&= K_\omega(0) + \int_0^t \left\{\mathrm{Im}\int n^\omega_t E^\omega \bar{E}^\omega_t + 6\int n^\omega_t |E^\omega_t|^2\right\},
\end{aligned} \tag{4.16}$$

where the constant $K_\omega(0)$ depends on the initial data and behaves at most like ω^2, since (4.10) implies $(1/\omega^2)|E^\omega_{tt}|_{L^2} \leq C$. Here and in the following, C as well as K and K' denote various constants whose unspecified values can change from one line to another line. Following the procedure developed in [8], we then obtain

$$\begin{aligned}
\left|\mathrm{Im}\int n^\omega_t E^\omega \bar{E}^\omega_t + 6\int n^\omega_t |E^\omega_t|^2\right| &\leq 6|n^\omega_t|_{H^{-1}}||E^\omega_t|^2|_{H^1} + |n^\omega_t|_{L^2}|E^\omega_t|_{L^2}|E^\omega|_{L^\infty} \\
&\leq K(|E^\omega_t|^2_{H^1} + |n^\omega_t|_{L^2}),
\end{aligned} \tag{4.17}$$

since $|n^\omega_t|_{H^{-1}}$ is controlled by $|V^\omega|_{L^2}$.

On the other hand, we estimate

$$2\left|\int (n_t^\omega(|E^\omega|^2)_t + n^\omega|E_t^\omega|^2)\right| \le 2(|n_t^\omega|_{H^{-1}}|\bar{E}^\omega E_t^\omega|_{H^1} + |n^\omega|_{L^2}|E_t^\omega|_{L^4}^2)$$

$$\le C + \tfrac{1}{2}|E_t^\omega|_{H^1}^2 + K|E_t^\omega|_{L^2}^2. \tag{4.18}$$

Eqs. (4.16)–(4.18) imply

$$|n_t^\omega|_{L^2}^2 + |V_t^\omega|_{L^2}^2 + \tfrac{1}{2}|E_t^\omega|_{H^1}^2 - \mathrm{Im}\left(\frac{1}{\omega^2}\int E_{tt}^\omega \bar{E}_t^\omega\right) + \frac{2}{\omega^2}|E_{tt}^\omega|_{L^2}^2$$

$$\le K_\omega(0) + C + K|E_t^\omega|_{L^2}^2 + \int_0^t C(|E_t^\omega|_{H^1}^2 + |n_t^\omega|_{L^2}^2). \tag{4.19}$$

In view of (4.14) we get

$$|E_t^\omega|_{L^2}^2 \le K + \mathrm{Im}\left(\frac{1}{\omega^2}\int E_{tt}^\omega \bar{E}_t^\omega\right) + \int_0^t C(|E_t^\omega|_{L^2}^2 + |n_t^\omega|_{L^2}^2),$$

and using this last inequality in (4.19), we find

$$|n_t^\omega|_{L^2}^2 + |V_t^\omega|_{L^2}^2 + \tfrac{1}{2}|E_t^\omega|_{H^1}^2 + \frac{2}{\omega^2}|E_{tt}^\omega|_{L^2}^2$$

$$\le K_\omega(0) + C + \frac{K'}{\omega^2}\mathrm{Im}\left(\int E_{tt}^\omega \bar{E}_t^\omega\right) + C\int_0^t (|E_t^\omega|_{H^1}^2 + |n_t^\omega|_{L^2}^2)$$

with

$$\left|\frac{K'}{\omega^2}\mathrm{Im}\left(\int E_{tt}^\omega \bar{E}_t^\omega\right)\right| \le K\left|\frac{E_t^\omega}{\omega}\right|_{L^2}\left|\frac{E_{tt}^\omega}{\omega}\right|_{L^2}$$

$$\le K + \frac{1}{\omega^2}|E_{tt}^\omega|_{L^2}^2,$$

which therefore yields

$$|n_t^\omega|_{L^2}^2 + |V_t^\omega|_{L^2}^2 + \tfrac{1}{2}|E_t^\omega|_{H^1}^2 + \frac{1}{\omega^2}|E_{tt}^\omega|_{L^2}^2 \le K_\omega(0) + C\int_0^t (|E_t^\omega|_{H^1}^2 + |n_t^\omega|_{L^2}^2).$$

The Gronwall's lemma finally implies that $|n_t^\omega|_{L^2}$, $|V_t^\omega|_{L^2}$, $|E_t^\omega|_{H^1}$ and $(1/\omega)|E_{tt}^\omega|_{L^2}$ are bounded, so that $|V^\omega|_{H^1}$, $|n^\omega|_{H^1}$ and $|E^\omega|_{H^2}$ can be bounded in turn by means of the basic equations (4.10)–(4.12). Note that all these bounds depend on ω (see the remark after the proof) but they are sufficient to prove the global existence of the solution for a fixed (even large) ω. In order to estimate $|V^\omega|_{H^2}$, $|n^\omega|_{H^2}$ and $|E^\omega|_{H^3}$, one has to perform again a calculation similar to the former one: we compute $\partial_{xx}(4.11) \times n_{xx}^\omega + \partial_{xx}(4.12) \times V_{xx}^\omega$, giving

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|n_{xx}^\omega|_{L^2}^2 + |V_{xx}^\omega|_{L^2}^2) = -c\int (|E^\omega|^2)_{xxx} V_{xx}^\omega, \tag{4.20}$$

and thereby the following estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|n_{xx}^\omega|_{L^2}^2 + |V_{xx}^\omega|_{L^2}^2) \le K\left(|E_{xxx}^\omega|_{L^2} + 1\right)|V_{xx}^\omega|_{L^2}. \tag{4.21}$$

On the other hand, $\partial_x(4.10) \times \bar{E}^{\omega}_{xxxt}$ leads to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\omega^2}|E^{\omega}_{xxt}|^2_{L^2} + |E^{\omega}_{xxx}|^2_{L^2}\right) = -2\mathrm{Re}\int (n^{\omega}E^{\omega})_{xx}\bar{E}^{\omega}_{xxt}$$
$$= -2\mathrm{Re}\int (n^{\omega}_{xx}E^{\omega} + 2n^{\omega}_{x}E^{\omega}_{x} + n^{\omega}E^{\omega}_{xx})\bar{E}^{\omega}_{xxt}, \qquad (4.22)$$

from which we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\omega^2}|E^{\omega}_{xxt}|^2_{L^2} + |E^{\omega}_{xxx}|^2_{L^2}\right) \le K(|n^{\omega}_{xx}|_{L^2} + 1)|E^{\omega}_{xxt}|_{L^2}. \qquad (4.23)$$

Summing up (4.21) and (4.23), and applying Gronwall's lemma yields the boundedness of $|n^{\omega}_{xx}|_{L^2}$, $|V^{\omega}_{xx}|_{L^2}$, $(1/\omega)|E^{\omega}_{xxt}|_{L^2}$ and $|E^{\omega}_{xxx}|_{L^2}$, and therefore ends the proof of Proposition 1. □

Remark 5. This kind of energy estimates may be used to construct an alternative proof of the existence theorem 1. Indeed, in order to obtain *local-in-time* estimates independent of ω, one has first to multiply $\partial_t(4.10)$ by $(1/\omega^2)\bar{E}^{\omega}_{tt}$, instead of $\bar{E}^{\omega}_{tt}$, then to iterate again the above procedure.

5. Conclusion

Summarizing our results, we have solved the Cauchy problem associated with the perturbed Zakharov equations (1.19) for a fixed value of c^2 and proved the convergence of the perturbed solution towards its time-enveloped limit. As displayed from the corrector expressions given in Theorem 2 and Corollary 2, it has moreover been proved that the time derivative of E^{ω} generally becomes oscillating when passing to the time envelope limit, unless the initial data satisfy some compatibility conditions imposed by the time-enveloped version of the Zakharov equations. Finally, we have shown the existence of unbounded solutions as $t \to \infty$ for not too large values of ω^2. Even though the latter proof only concerns some infinite-time blow-up, it can be emphasized that this singular dynamics could yield a strong indication in favor of the fact that, from a physical viewpoint, Langmuir wave-packets may catastrophically evolve towards collapse in a finite time (this assertion remaining of course the salient open question owing to the mathematical treatment of the strong Langmuir turbulence). Following the hypotheses (C1) and (C2) of the last theorem, such singular solutions should physically develop for a strong coupling between the large-scale cavities and the Langmuir envelope. When looking at Langmuir envelopes whose frequency remains rather close to the plasma electron one, this situation could indeed concern negative-energy states satisfying $\mathcal{E}_{\omega} < 0$ as the charge Q_{ω} is assured to be positive in the limit $\omega^2 >> 1$. In this situation, one can expect that the "mass" density $|E^{\omega}|^2$ will rapidly increase, thus enhance the ponderomotive force, and finally drive the ion cavities towards a strongly supersonic evolution where Landau damping is awaited for saturating this sudden growth of plasma waves. Before this ultimate stage, the proper mass of the trapped Langmuir waves – corresponding here to their L^2 norm – does not remain invariant and can even strongly increase in time, unlike the time-enveloped version of the Zakharov equations for which the same norm is a constant of motion. As already encountered in the scope of the nonlinear Schrödinger equation [4], the "residual" variations induced by the electron oscillations acting on the Langmuir envelope thus participate to the blow-up dynamics, and compared with the standard Zakharov prescription, they could be thought to accelerate the collapse process developing within the inertial range of the SLT. To clear up this point, we should nevertheless understand how the shape and the time scales of a blow-up described by the perturbed Zakharov equations can fit with the ones associated with the singular solutions to the usual Zakharov equations in the limit of large ω, since the characteristic time scale for collapse may differ passing from the perturbed model to the usual one. At the present state, this question, however, remains under investigation.

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Appendix A

As evoked in the introductory part of this paper, a difficult mathematical problem arises when a subsonic limit is forced on both equations of motion, i.e. when we simultaneously pass to the limits $c^2 \to \infty$ and $\omega^2 \to \infty$ in system (2.1) and (2.2). The main difficulties that hinder to answer this question can be summarized as follows : let us first recall that Schochet and Weinstein [10] addressed the case when $c^2 \to \infty$ in the unperturbed Zakharov equations

$$\mathrm{i}E_t + \Delta E = nE,$$
$$\frac{1}{c^2}n_{tt} - \Delta n = \Delta|E|^2,$$

about which they proved the convergence to the nonlinear Schrödinger equation by transforming this system into the dispersive perturbation of a symmetric hyperbolic one. Nevertheless, in the present situation, this transformation does not give a symmetric system when nE is replaced by $-nE$ in the first equation. Concerning (3.1) and (3.2), the decomposition of E into F and G indeed shows that both signs in front of the nonlinearity appear (see (2.14) and (2.15)), so that Schochet–Weinstein's method does not apply any longer. In addition, our results cannot apply to the limits $\omega^2 \to \infty$, $c^2 \to \infty$ because our proofs are not uniform with respect to c. Therefore it seems impossible to couple directly our asymptotics with Schochet–Weinstein's one.

On the other hand, the "adiabatic" approximation on (2.1) and (2.2), which consists in taking the limit of less physical significance $c^2 \to \infty$ while ω^2 is kept fixed, can be treated in the following way (arrows now concern the latter limit) :

Theorem 4. Let us consider the system

$$\frac{1}{\omega^2}\frac{\partial^2 E^c}{\partial t^2} - 2\mathrm{i}\frac{\partial E^c}{\partial t} - \Delta E^c = -n^c E^c, \tag{A.1}$$

$$\frac{1}{c^2}\frac{\partial^2 n^c}{\partial t^2} - \Delta n^c = \Delta|E^c|^2 \tag{A.2}$$

with initial data $E^c(0) = E_0 \in H^{s+1}$, $E^c_t(0) = E_1 \in H^s$, $n^c(0) = n_0 \in H^s$ and $n^c_t(0) = n_1 \in H^{s-1}$ for $s > \frac{1}{2}d$. Then there exists $T > 0$ (independent of c) such that there exists a unique solution (E^c, n^c) to (A.1) and (A.2) satisfying

$$(E^c, E^c_t, n^c, n^c_t/c) \in C([0,T], H^{s+1} \times H^s \times H^s \times H^{s+1}).$$

Let T^c be the existence time of $(E^c, E^c_t, n^c, n^c_t/c)$ and T^∞ be the existence time of the solution to

$$\frac{1}{\omega^2}\frac{\partial^2 E}{\partial t^2} - 2\mathrm{i}\frac{\partial E}{\partial t} - \Delta E = |E|^2 E, \tag{A.3}$$

$$E(0) = E_0, \quad \frac{\partial E}{\partial t}(0) = E_1. \tag{A.4}$$

Then

$$\liminf_{c\to\infty} T^c \geq T^\infty,$$

and for all $T_1 < T^\infty$ the following convergences hold as $c^2 \to \infty$:

$$E^c \to E \text{ in } L^\infty(0, T_1; H^{s+1}),$$

$$E_t^c \to E_t \text{ in } L^\infty(0, T_1; H^s),$$

$$n^c + |E^c|^2 - \cos(c(-\Delta)^{1/2}t)(n_0 + |E_0|^2) - \frac{1}{c}\sin(c(-\Delta)^{1/2}t)(-\Delta)^{-1/2}(n_1 + 2\mathrm{Re}(\bar{E}_0 E_1)) \to 0 \text{ in } L^\infty(0, T_1; H^s),$$

$$\frac{1}{c}n_t^c - (-\Delta)^{1/2}\sin(c(-\Delta)^{1/2}t)n_0 - \cos(c(-\Delta)^{1/2}t)\frac{n_1}{c} \to 0 \text{ in } L^\infty(0, T_1; H^{s-1}).$$

Proof (sketch)

We use again the transformation (2.6) and we introduce $P^c = P/c^2$. Eqs. (A.1) and (A.2) become

$$\frac{1}{\omega^2}\frac{\partial^2 E^c}{\partial t^2} - 2\mathrm{i}\frac{\partial E^c}{\partial t} - \Delta E^c = -\frac{2\omega^2 c^2}{c^2-\omega^2}P^c E^c + \frac{c^2}{c^2-\omega^2}|E^c|^2 E^c, \tag{A.5}$$

$$\frac{\partial^2 P^c}{\partial t^2} - c^2\Delta P^c = \frac{1}{\omega^2}\left|\frac{\partial E^c}{\partial t}\right|^2 - 2\mathrm{Im}\left(\frac{\partial E^c}{\partial t}\bar{E}^c\right) - |\nabla E^c|^2 - \frac{2\omega^2 c^2}{c^2-\omega^2}P^c|E^c|^2 + \frac{c^2}{c^2-\omega^2}|E^c|^4, \tag{A.6}$$

and therefore lead to the following integral system

$$E^c = S_o^\omega(t)E_0 + S_1^\omega(t)E_1 + \omega^2\int_0^t S_1^\omega(t-s)\left[-\frac{2\omega^2 c^2}{c^2-\omega^2}P^c E^c + \frac{c^2}{c^2-\omega^2}|E^c|^2 E_c\right](s)\,\mathrm{d}s, \tag{A.7}$$

$$\begin{aligned} P^c = {} & \cos(c(-\Delta)^{1/2}t)P_0^c + \frac{1}{c}\sin(c(-\Delta)^{1/2}t)(-\Delta)^{-1/2}P_1^c \\ & + \int_0^t \frac{1}{c}\sin(c(-\Delta)^{1/2}(t-s))(-\Delta)^{-1/2} \\ & \times\left[\frac{1}{\omega^2}\left|\frac{\partial E^c}{\partial t}\right|^2 - 2\mathrm{Im}\left(\frac{\partial E^c}{\partial t}\bar{E}^c\right) - |\nabla E^c|^2 - \frac{2\omega^2 c^2}{c^2-\omega^2}P^c|E^c|^2 + \frac{c^2}{c^2-\omega^2}|E^c|^4\right](s)\,\mathrm{d}s. \end{aligned} \tag{A.8}$$

Using the same techniques as those employed in Sections 2 and 3, we then obtain the results of Theorem 4. □

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