PH.D. PROPOSAL: DISTRIBUTIONALLY ROBUST SHAPE OPTIMIZATION

In a nutshell, the proposed Ph.D. project aims to solve shape optimization problems using the recent distributionally robust optimization methods combined with Wasserstein distances.

Shape and topology optimization. Shape and topology optimization is about finding the "best" shape of a domain Ω with respect to a given performance criterion, under some constraints. In the recent decades, the great headway made in the numerical realization of such programs have aroused a tremendous enthusiasm in both academic and industrial worlds, see [3] for a comprehensive presentation of the subject and Fig. 1 for a couple of examples. From the mathematical viewpoint, shape optimization problems typically show up under the following generic form:

$$(\mathcal{P}) \qquad \min_{\Omega} \mathcal{C}(\Omega, f),$$

where

- The optimized shape Ω is a bounded domain in $\mathbb{R}^d$, representing a physical system, for instance a mechanical structure, a fluid device, etc.,
- The parameters f correspond to physical data such as applied forces, material parameters, etc.,
- The cost function $\mathcal{C}(\Omega, f)$ measures the physical performance of Ω under the operating conditions characterized by f,

and where constraints are omitted for simplicity of the exposition.

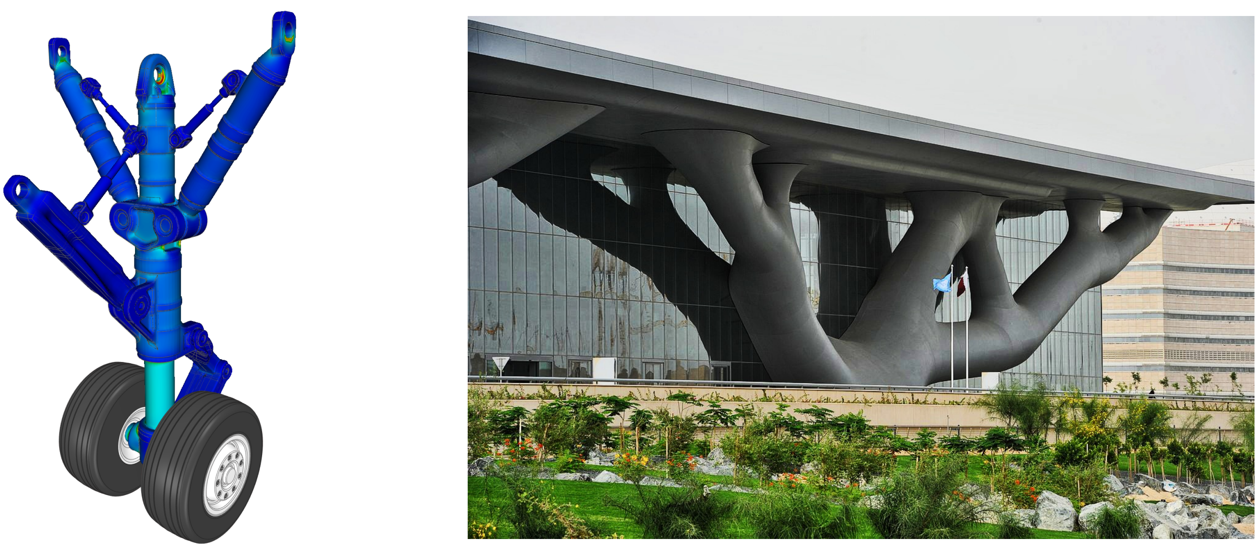

FIGURE 1. *(Left) Shape and topology optimization of a landing gear by an industrial software (courtesy of* `Ansys`*) (right) The Qatar National Convention Center (architect: Arata Isozaki) has the typical outline of a topology optimized structure (licence Creative Commons)*

Sensibility to data measurements. In most applications, $\mathcal{C}(\Omega, f)$ depends on Ω and f via a "state" function $u_{\Omega,f}$, describing its physical behavior. For instance, $u_{\Omega,f}$ may stand for the elastic displacement of an elastic structure Ω submitted to traction loads f, or the velocity of a fluid conveyed by a pipe Ω, subject to gravity forces f, etc. From the mathematical viewpoint, $u_{\Omega,f}$ arises as the solution to a partial differential equation posed on Ω, where f plays the role of a parameter.

Unfortunately, these data are at best measured or estimated, and thereby known up to some uncertainty. On the other hand, the physical behavior $u_{\Omega,f}$ of the device and its physical performance $\mathcal{C}(\Omega, f)$ are very sensitive to even small perturbations on these data. This raises the need to incorporate a degree of awareness of such uncertainties into shape optimization problems. This task usually fits in one of the following two frameworks, see e.g. [8] for an overview:

- In situations where no information is available about the data f except for an upper bound m on their amplitude $||f||$, worst-case design approaches are often retained. In a nutshell, these aim to minimize the worst value of the cost function $\mathcal{C}(\Omega, f)$ among all possible data functions, that is:
$$\min_{\Omega} J_{\text{wc}}(\Omega), \text{ where } J_{\text{wc}}(\Omega) := \sup_{||f|| \leq m} \mathcal{C}(\Omega, f).$$
We refer e.g. to [4, 1] for associated references in the field of optimal design.
- When the law $\mathbb{P}$ of the unknown parameter f is available, probabilistic approaches aim to minimize the mean value (or another quantile) of the cost function $\mathcal{C}(\Omega, f)$, or a probability of failure of the device; namely, objective functions of the domain of the following form are considered:
$$J_{\text{mean}}(\Omega) = \mathbb{E}_{f\sim\mathbb{P}}\left[\mathcal{C}(\Omega, f)\right], \text{ or } J_{\text{rel}}(\Omega) = \mathbb{P}\left(\mathcal{C}(\Omega, f) \leq \alpha\right),$$
where α is a safety threshold for the system; see for instance [2, 5].

Unfortunately, both settings suffer from major drawbacks. Beyond the tremendous practical difficulties posed by worst-case design approaches, these often turn out to be "too pessimistic", insofar as they tend to produce optimized designs Ω with poor nominal performance for the sake of accommodating worst-case situations which are very unlikely to occur in practice. On the other hand, probabilistic approaches fundamentally rely on the knowledge of the distribution of the random parameter f, which is only available through (costly) observations and statistics.

Distributionally robust shape optimization. The idea of *distributionally robust optimization* has recently emerged in the field of stochastic and convex programming as a tentative remedy for the aforementioned weakness of probabilistic approaches, see [11] for an overview. Here, we apply this method in the context of shape optimization: Retaining the jargon of shape optimization, this paradigm relies on the datum of a "nominal" probability distribution $\mathbb{P}$ for the uncertain parameter f, which is typically constructed as the empirical mean of a series of observed values $f_1, \ldots, f_N$ for f:

$$\mathbb{P} = \frac{1}{N}\sum_{i=1}^{N} \delta_{f_i}.$$

Then, one aims to optimize the worst value of the expected cost $\mathbb{E}_{f\sim\mathbb{Q}}\left[\mathcal{C}(\Omega, f)\right]$ when f is governed by a probability law $\mathbb{Q}$ which is "close" to $\mathbb{P}$. In other terms, the distributionally robust shape optimization problem reads:

$$(\mathcal{P}_{\text{ds}}) \qquad \min_{\Omega} J_{\text{ds}}(\Omega), \text{ where } J_{\text{ds}}(\Omega) = \sup_{d(\mathbb{Q},\mathbb{P})\leq m} \mathbb{E}_{f\sim\mathbb{Q}}\left[\mathcal{C}(\Omega, f)\right],$$

where $d(\cdot,\cdot)$ is a distance acting on probability measures, and m is the allowed maximum discrepancy between the actual and nominal laws $\mathbb{Q}$ and $\mathbb{P}$.

Optimal transport distance. A key feature of distributionally robust optimization problems of the form $(\mathcal{P}_{\text{ds}})$ is the choice of an adapted measure of distance $d(\mathbb{Q},\mathbb{P})$ between probability measures. Hitherto, most studies have focused on "pointwise" notions of distance, such as the Kullback-Leibler divergence, mainly out of simplicity. Very recently, attempts have been made to use more "faithful" notions such as Wasserstein distances, pertaining to the optimal transport theory, see [10, 12, 9] about optimal transport, and [6, 7] about the use of Wasserstein distances in distributionally robust optimization.

Goals and challenges of the Ph.D. project. The main purpose of this Ph.D. proposal is to address shape and topology optimization problems using distributionally robust optimization and optimal transport based distances. To be more precise, we consider the problem $(\mathcal{P}_{\text{ds}})$ where $d(\cdot,\cdot)$ is a Wasserstein distance. The implementation of this program raises several difficulties:

- The distributionally robust optimization paradigm is new, and it has only been applied in academic situations, featuring a finite-dimensional optimization variable (and not a shape Ω) and random parameter f. They feature strong assumptions on the cost function (such as convexity with respect to the optimization variable, concavity with respect to the uncertain parameter) which are rarely met in practice. Their extension to the shape optimization setting and the device of practical algorithms in this context is a challenging issue.
- The use of Wasserstein distance also brings numerical issues, and leveraging recent achievements in the field of optimal transport [10, 9] will be required to devise convenient and tractable reformulations of the problem, at least in some model cases (e.g. when the cost function $\mathcal{C}(\Omega, f)$ is the elastic energy, or when the state $u_{\Omega,f}$ has a linear behavior with respect to the uncertain parameter f).

The scope of this Ph.D. project is theoretical as well as numerical:

- On the one hand, we aim at developing tractable numerical methods for the optimal design of shapes when uncertainties over the physical data are accounted for in the distributionally robust optimization framework. The theoretical fundations of the devised models will be investigated, with the ambition to achieve "benchmark" theoretical guarantees, with restrictive and academic underlying assumptions, supplying qualitative and formal insights about the relevance of the models.
- On the other hand, we aim to implement the proposed distributionally robust shape optimization methods, at first in the context of academic, benchmark problems (relying, for instance, on the simplifying density-based optimization setting), then in the context of quite realistic optimal design problems in structural mechanics (optimization of an electric pylon or a crane under uncertain wind conditions, optimization of the shape of a bridge under uncertain car traffic, etc.).

Hence, the Ph.D. student is expected to have a good theoretical background as well as some programming skills. A basic knowledge of the fields of shape optimization or optimal transport would be a valuable asset.

Contact. The Ph.D. project is funded by a scholarship for Université Grenoble Alpes. It unfolds in Laboratoire Jean Kuntzmann, starting from October 2022. It is jointly supervised by Charles Dapogny[1] and Boris Thibert[2] . Applications must be sent by email together with a detailed curriculum vitae.

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