Geodesic as limits of geodesics on PL-surfaces

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Abstract

In this paper, we study the problem of convergence of geodesics on PL-surfaces and in particular on subdivision surfaces. More precisely, if a sequence $(T_n)_{n\in\mathbb{N}}$ of PL-surfaces converges in distance and in normals to a smooth surface S and if C_n is a geodesic of T_n (*i.e.* it is locally a shortest path) such that $(C_n)_{n\in\mathbb{N}}$ converges to a curve C, we want to know if the limit curve C is a geodesic of S. Hildebrandt et al. [12] have already shown that if C_n is a shortest path, then C is also a shortest path. The result does not hold anymore for geodesics that are not (global) shortest paths. In this paper, we first provide a counter example for geodesics: we build a sequence $(T_n)_{n\in\mathbb{N}}$ of PL-surfaces that converges in distance and in normals to the plane. On each T_n, we build a geodesic C_n, such that $(C_n)_{n\in\mathbb{N}}$ converges to a planar curve which is not a line-segment (and thus not a geodesic of the plane). In a second step, we give a positive result of convergence for geodesics that needs additional assumptions concerning the rate of convergence of the normals and of the lengths of the edges of the PL-surfaces. Finally, we apply this result to different subdivisions surfaces (following schemes for bicubic B-splines, or Catmull-Clark schemes, or schemes for Bezier surfaces). In particular, these results validate an algorithm of Pham-Trong et al. [20] that builds geodesics on subdivision surfaces.

Keywords : subdivision surfaces, triangulations, PL-surfaces, geodesics, shortest paths, convergence

1 Introduction

A geodesic is usually defined as a curve on a surface that is locally a shortest path. Geodesics appear naturally in several applications, among which we can mention: i) The modelling of the human heart: the heart left ventricle can be modelled by a family of embedded surfaces; a muscular fiber of the central region of the left ventricle has

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particular properties and can be considered as a geodesic of one of those surfaces [17, 21]. ii) In the fabrication of composite parts by filament winding, the filament must idealy wind along geodesics [1]. iii) Finally the computation of radar cross sections involves the simulation of creeping ray which follow geodesics of the object [2, 5]. In this context, and since piecewise linear 2-manifolds (denoted by PL-surfaces in the following) are widely used for surface modelling, it is natural to consider the modelling of geodesics on surfaces and in particular on PL-surfaces.

We distinguish the geodesics from the more restricted class of shortest paths. A shortest path is a curve on a surface that is connecting two points and whose length does not decrease if it is perturbed (without moving the two extremities). A geodesic is a curve on a surface whose length does not decrease if it is pertubed in a small neighborhood of any point. A shortest path is clearly a geodesic, but the converse is not true (for example, a great circle is a geodesic but not a shortest path of the sphere).

There exist several algorithms that build shortest paths on PL-surfaces [13, 14, 15, 19]. Concerning the geodesics, Pham-Trong and her coauthors have also proposed an algorithm that builds geodesics on PL-surfaces [20]. In particular, they have also considered a sequence $(T_n)_{n\in\mathbb{N}}$ of PL-surfaces defined by the De Casteljau subdivision for Bezier surfaces that is converging to a Bezier surface S. On each T_n, they build a geodesic C_n whose sequence converges to a curve C. The natural question is then to wonder whether C is a geodesic of S.

The convergence of geodesics has already been studied in the case of shortest paths by Hildebrandt et al. [12] and Memoli et al. [16]. They show that if a sequence $(T_n)_{n\in\mathbb{N}}$ of PL-surfaces converges in Hausdorff distance to S, if the normals of T_n also converge to the normals of S, then the limit curve of a sequence of shortest paths is a shortest path of S. However, this result does not hold anymore for geodesics: we provide in this paper (Section 4) a sequence $(T_n)_{n\in\mathbb{N}}$ of PL-surfaces whose both distance and angular limit is a plane. However a sequence of geodesics $C_n \subset T_n$ converges toward a limit curve C which is not a straight line of the plane, and thus not a geodesic of the plane.

It is worth noting that the result of convergence of Hildebrandt et al. [12] cannot be used in some applications: for example, in the modelling of the human heart, the curves modelling the fibers are closed and are not shortest paths [17]. Furthermore, this result cannot be used to validate the algorithm given in [20]: indeed, Pham Trong and her coauthors build a sequence of geodesics that are not shortest paths in general.

The main result of this paper deals with convergence for geodesics. More precisely, we suppose, as for the result with shortest paths given in [12], that the sequence $(T_n)_{n\in\mathbb{N}}$ of PL-surfaces converges in Hausdorff distance and in normals to a smooth surface S. In addition, we also suppose that there exist two constants K_1 and K_2 independant on n such that the length of the edges of T_n is greater than $\frac{K_1}{2^n}$ and the maximal angle between the normals of T_n and the normals of S is less than $\frac{K_2}{2^n}$. In other words, the rate of convergence of the length of the edges cannot be faster than the rate of convergence of the normals.

This result is then applied to PL-surfaces T_n that follow subdivision schemes, such as for example schemes for bicubic B-splines, or Catmull-Clark schemes, or Bezier surfaces. In particular, our results validate the algorithm of Pham-Trong and her coauthors [20] that builds geodesics on subdivision surfaces.

It is interesting to note that shortest paths and geodesics do not deal with notions of the same order: the notion of shortest path relies on the notion of length, which is a quantity related to the first derivative. However, since a geodesic is defined locally, it depends on the infinitesimal variation of the length, which is a notion of second order.

In this paper, we focus on the problem of convergence of geodesics. In Section 2, we recall the main definitions. In Section 3, we recall the result of convergence of [12, 16] for shortest paths. In Section 4 we give a counter example showing that the situation is more complicated for geodesics, and we also give the main result of convergence. We show in Section 5 that this result can be applied to several subdivision schemes (and in particular to the algorithm proposed in [20]). The last section proves all these results. In the following, we will refer to triangulations instead of PL-surfaces.

2 Definitions

2.1 Smooth surfaces

In the following, a smooth surface means a $\mathcal{C}^2$ surface which is regular, oriented, compact with or without boundary. We have the following proposition [11]

Proposition 1 *Let S be a smooth compact surface of $\mathbb{R}^3$. Then there exists an open set U_S of $\mathbb{R}^3$ containing S and a continuous map ξ from U_S onto S satisfying the following: if p belongs to U_S, then there exists a unique point $\xi(p)$ realizing the distance from p to S (ξ is nothing but the orthogonal projection onto S).*

This proposition allows to introduce the following notion introduced by H. Federer [11]: The *reach of a surface* S is the largest $r > 0$ for which ξ is defined on the open tubular neighborhood $U_r(S)$ of radius r of S.

2.2 Triangulations

A triangulation T is a connected topological 2-manifold made of a finite union of triangles of $\mathbb{R}^3$, such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge.

2.3 Curves

In the following a curve C means a lipschitz parametrized curve $C : [0,1] \to \mathbb{R}^3$. Its length is denoted by $l(C)$. Similarly, for $0 \leq t_a <$

$t_b \leq 1$ we denote by $l(C, t_a, t_b)$ the length of the curve C restricted to $[t_a, t_b]$. As a particular case of Rademacher theorem, we know that C is differentiable almost everywhere. Moreover, it is the integral of its derivative. Whenever it exists, we denote by $C'(t)$ the derivative of C at t.
• We say that C has a *uniform parametrization* if it satisfies for almost every $t \in [0,1]$ $\|\mathcal{C}'(t)\| = l(C)$.
• A curve $C : [0,1] \to M \subset \mathbb{R}^3$ with uniform parametrization is said to be a geodesic of a lipschitz surface M (M can be a triangulation or a smooth surface) if it locally minimizes the length, *i.e.* if for every $t \in [0,1]$, there exists $0 \leq t_a \leq t \leq t_b \leq 1$ (where $t_a < t$ if $t > 0$ and $t_b > t$ if $t < 1$), such that any lipschitz curve $\widetilde{C} : [0,1] \to M$ such that $\widetilde{C}(0) = C(t_a)$ and $\widetilde{C}(1) = C(t_b)$ satisfies

$$l(C, t_a, t_b) \leq l(\widetilde{C}).$$

The geodesic is said to be interior if for every $t \in [0,1]$, $C(t)$ is interior to the surface M. The geodesic C is a shortest path, if the length of any curve on M connecting $C(0)$ and $C(1)$ has a length greater than $l(C)$.

2.4 Properties of geodesics on triangulations

Let C be a polygonal curve of a triangulation T. Then C is a geodesic of T if and only if:

- C it is a straight line on each triangle. (The vertices p of C then belongs to the edges of T or are vertices of T.)
- If p belongs to the interior of an edge of T, then the incident and refracted angles of C at p are equal (see Figure 1).
- If p is a vertex of T, then C separates the set of the triangles of T containing p into two connected regions r_1 and r_2 (see Figure 1). If one denotes by $\alpha_p^{r_1}$ the sum of the angles $\alpha_i^{r_1}$ of the triangles of region r_1 at p (*resp.* by $\alpha_p^{r_2}$ the angles $\alpha_i^{r_2}$ of the triangles of region r_2 at p), one has

$$\alpha_p^{r_1} \geq \pi \quad \text{and} \quad \alpha_p^{r_2} \geq \pi. \tag{1}$$

Remark that if a geodesic traverses a vertex p of T, then the sum of the angles of the triangles of T at the vertex p is greater than 2π: $\alpha_p^{r_1} + \alpha_p^{r_2} \geq 2\pi$.

We can also notice that if $\alpha_p^{r_1} + \alpha_p^{r_2} > 2\pi$, then the geodesic containing the vertices q and p is not unique (see Figure 1): two distinct polygonal curves C_1 and C_2 containing p and q that satisfy Equation (1) are geodesics.

3 Convergence of shortest paths

In this section, we recall a positive result of convergence for shortest paths, that can be found in [12] (a similar result can also be found in

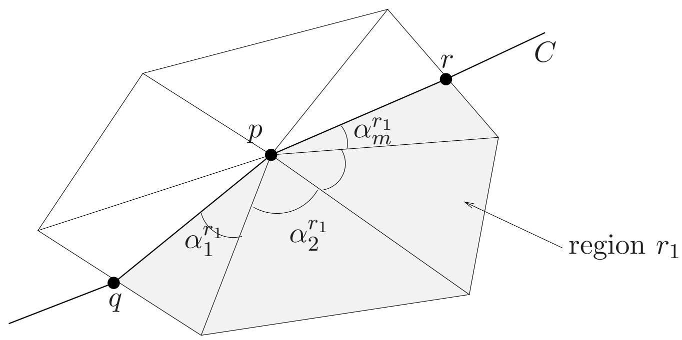

Figure 1: A geodesic C passing through a vertex of a $3D$ triangulation

[16]). Proposition 2 states that if a sequence $(T_n)_{n\in\mathbb{N}}$ of triangulations tends to a surface S in distance and in normals, then a sequence of shortest paths of T_n is converging to a shortest path of S. Although this proposition has already been proved, for the sake of completeness, we give here a proof.

Proposition 2 *Let $(T_n)_{n\in\mathbb{N}}$ be sequence of triangulations that converges in the Hausdorff sense to a smooth $\mathcal{C}^2$ surface S. Let $(d_n)_{n\in\mathbb{N}}$ be sequence of real numbers converging to* 0. *We suppose that*

a) for every n, the restriction $\xi_n : T_n \to S$ of the map ξ to T_n is bijective;

b) for every $m \in T_n$, the angle between any triangle Δ containing m and the tangent plane $\Pi^S_{\xi(m)}$ of S at $\xi(m)$ is smaller than d_n;

Let a and b be two points of S, and $C_n : [0,1] \to \mathbb{R}^3$ be a shortest path on T_n between $\xi_n^{-1}(a)$ and $\xi_n^{-1}(b)$, with uniform parametrization. Then

- *there exists a subsequence (C_{n_k}) of (C_n) that is uniformly converging to a curve C;*
- *if a subsequence of (C_n) is converging to a limit curve C, then C is a shortest path of S.*

Proof of Proposition 2
First remark that the convergence in distance and in normals imply that there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0 such that for every curve γ_n of T_n we have (see [18] or [12]):

$$(1-\epsilon_n)l(\gamma_n) \le l(\xi_n \circ \gamma_n) \le (1+\epsilon_n)l(\gamma_n).$$

Let us denote by $\widetilde{C}$ a shortest path between a and b on S. We then have :

$$l(\xi_n \circ C_n) \le (1+\epsilon_n)l(C_n) \le (1+\epsilon_n)l(\xi_n^{-1}\circ \widetilde{C}) \le \frac{1+\epsilon_n}{1-\epsilon_n} l(\widetilde{C}),$$

which finally gives

$$l(\widetilde{C}) \le l(\xi_n \circ C_n) \le \frac{1+\epsilon_n}{1-\epsilon_n} l(\widetilde{C}). \tag{2}$$

In particular $l(\xi_n \circ C_n)$ converges to $l(\widetilde{C})$ and is then bounded. Then there exists a constant k independant of n such that for every $t, t' \in (0,1)$

$$\begin{aligned} \|\xi_n \circ C_n(t) - \xi_n \circ C_n(t')\| & \leq l(\xi_n \circ C_n, t, t') \\ & \leq (1+\epsilon_n)\ l(C_n, t, t') \\ & = (1+\epsilon_n)\ l(C_n)|t-t'| \\ & \leq k\ |t-t'|. \end{aligned}$$

The set $\{\xi_n \circ C_n,\ n \in \mathbb{N}\}$ is then equicontinuous (see [6] for example for more details). Arzela-Ascoli Theorem [6] then implies that a subsequence $(\xi_{n_k} \circ C_{n_k})_{k\in\mathbb{N}}$ of $(\xi_n \circ C_n)_{n\in\mathbb{N}}$ is uniformly converging to a curve C. Remark now that for every $t \in (0,1)$, the point $\xi_n(C_n(t))$ is the closest point of S to $C_n(t)$. That implies that $\|C_n(t) - \xi_n \circ C_n(t)\|$ is smaller than the Hausdorff distance between T_n and S, which tends to 0. Therefore, the subsequence $(C_{n_k})_{k\in\mathbb{N}}$ is also uniformly converging to C.

The fact that C_n uniformly converges to C implies that $l(C) \leq \liminf l(C_n)$. However, Equation (2) implies that $\lim C_n = \lim \xi_n \circ C_n = l(\widetilde{C})$. We then have $l(C) \leq l(\widetilde{C})$ which means that C is a shortest path. □

4 Convergence of geodesics

The main result of this paper is given in this section. We first show in Section 4.1, by building a counter-example, that the previous result of convergence for shortest paths, does not hold anymore for geodesics in general. Then, in Section 4.2, we give a general result of convergence for geodesics.

4.1 Counter-example

The counter example shows a sequence $(T_n)_{n\in\mathbb{N}}$ of triangulations whose both distance and angular limit is a plane. However a sequence of geodesics $C_n \subset T_n$ converges toward a limit curve C which is not a straight line of the plane, and thus not a geodesic of the plane. We can notice that the triangulations T_n and the curves C_n satisfy all the assumptions of the result of Proposition 2, except that C_n is a geodesic (and not a shortest path). However the limit curve C is not a geodesic. This counter example implies that the convergence in distance and in normals of T_n to S is not sufficient to expect a result of convergence for geodesic.

Detail of the construction:

Figure 2 shows the triangulation T_n for some n. The triangulation overlaps with the horizontal plane Π_H of equation $z = 0$ outside the large circle and inside the small one. In the ring between the two circles, it is made of 4^n identical small "roof shaped" bumps detailed on the right of Figure 2. The points d_n^1, d_n^2, p_n and m_n are on the plane Π_H while the points t_n^1 and t_n^2 stand at some height above the

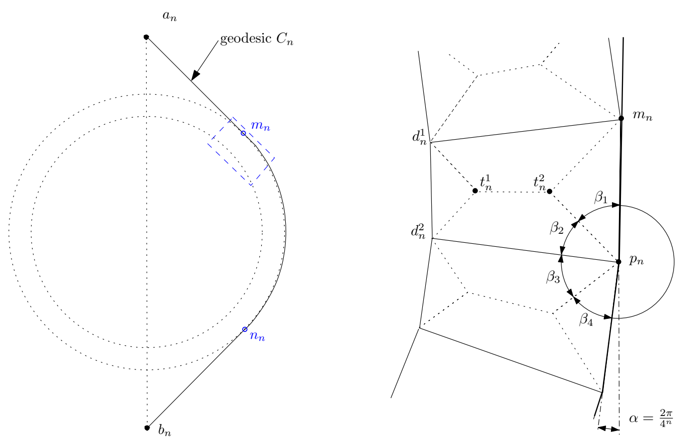

Figure 2: Triangulation T_n and geodesic C_n seen from above: on the left we see the whole surface; the region in the dashed rectangle is depicted on the right

plane. The faces $d_n^1 t_n^1 d_n^2$, $p_n t_n^2 m_n$, $d_n^1 t_n^1 t_n^2 m_n$ and $d_n^2 t_n^1 t_n^2 p_n$ are planar and all make a slope s_n with Π_H. If we take $s_n = \frac{3}{2^n}$, one has, for each $n \in \mathbb{N}$, $\beta_1 + \beta_2 + \beta_3 + \beta_4 > \pi$.

Observe that the sequence $(T_n)_{n\in\mathbb{N}}$ of triangulations converges toward the plane in the Hausdorff sense. Furthermore the normals of T_n tend to the normals of the plane. The shortest path on T_n between the point a_n and b_n is the straight line (dotted line on Figure 2). However, the line $a_n m_n n_n b_n$, wraped around the circle between m_n and n_n is a local minimum, that is a geodesic, between a_n and b_n (see Section 2). These geodesics converge in the Hausdorff sense toward a curve C composed of two lines and an arc of circle.

4.2 Convergence toward a geodesic

The main result of this paper is the following theorem. It states that if a sequence $(T_n)_{n\in\mathbb{N}}$ of triangulations is converging to a smooth surface S, then, under reasonable assumptions, a sequence of geodesics of T_n is converging to a geodesic of S.

Theorem 1 *Let S be a smooth surface of $\mathbb{R}^3$, r denote the reach of S, and (T_n) be a sequence of triangulations. Let K, $\widetilde{K}$, θ_{min} be positive constants and (d_n) a sequence of real numbers converging to 0. Suppose that for every n:*

a) T_n belongs to the tubular neighborhood $U_r(S)$ of radius r of S;

b) for every $m \in T_n$, $\|m - \xi(m)\| \le d_n$;

c) for every $m \in T_n$, the angle between any triangle Δ containing m and the tangent plane $\Pi^S_{\xi(m)}$ of S at $\xi(m)$ is smaller than $\frac{K}{2^n}$;

d) the lengths of the edges of T_n are greater than $\frac{\widetilde{K}}{2^n}$;

e) all the angles of T_n are greater than θ_{min};

Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of polygonal curves $C_n : [0,1] \to \mathbb{R}^3$ with uniform parametrization such that C_n is an interior geodesic of T_n and $\xi(C_n)$ does not intersect the boundary of S. If $(C_n)_{n\in\mathbb{N}}$ converges toward a curve C in the sup norm sense, then C is of class $\mathcal{C}^2$ and is a geodesic of S.

The proof of this theorem is given in Section 6.

Remark 1 If a sequence $(T_n)_{n\in\mathbb{N}}$ of triangulations satisfies the conditions of Theorem 1, then the valency of the vertices of T_n is uniformly bounded by a constant independant on n.
Indeed, the convergence of the normals implies that the sum of the angles at a vertex p_n tends to 2π when n tends to infinity. Combined with the fact that all the angles are greater than θ_{min}, that implies that the valency of p_n is uniformly bounded.

Remark 2 Remark that the result of Theorem 1 still holds if the sequence $(C_n)_{n\in\mathbb{N}}$ does not converge to a curve C, but if we suppose that $(l(C_n))_{n\in\mathbb{N}}$ is bounded. Indeed, in that case, we can show (as in the proof of Proposition 2) that the family $\{C_n,\ n\in\mathbb{N}\}$ is equicontinuous. The Arzela-Ascoli theorem (see [6] for example) then implies that a subsequence $(C_{n_k})_{k\in\mathbb{N}}$ of $(C_n)_{n\in\mathbb{N}}$ uniformly converges to a curve C. Theorem 1 can then be applied to $(C_{n_k})_{k\in\mathbb{N}}$.

5 Application to subdivision surfaces

The previous theorem can be easily applied to subdivision surfaces. In this section, we first give a general corollary, Corollary 1, that can be easily applied to several subdivision schemes. We then show that the result of convergence for geodesics holds when the triangulations follow a subdivision scheme for either bicubic B-splines, or Catmull-Clark schemes, or Bezier surfaces. We first need to give a few definitions.

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of parametrized triangulations $P_n : [0,1]^2 \to \mathbb{R}^3$ that is converging to a paramatrized smooth surface $f : [0,1]^2 \to \mathbb{R}^3$. The parameter domain $[0,1]^2$ of each P_n can be triangulated so that P_n is linear on each triangle of $[0,1]^2$.
– We say that the parameter domain of P_n is a triangular grid if its vertices are $p_n^{i,j} = \left(\frac{i}{2^n}, \frac{j}{2^n}\right)$ (where $i,j \in \{0,..2^n\}$) and the edges are $p_n^{i,j}p_n^{i+1,j}$, $p_n^{i,j}p_n^{i,j+1}$ and $p_n^{i,j}p_n^{i+1,j+1}$.
– We say that $(P_n)_{n\in\mathbb{N}}$ uniformly converges to a function f with rate of convergence $\frac{1}{2^n}$ if:

$$\exists N\in\mathbb{N},\ \exists K\in\mathbb{R},\ n>N \ \Rightarrow \sup_{(u,v)\in[0,1]^2} \|P_n(u,v)-f(u,v)\| \le \frac{K}{2^n}.$$

– We say that $(P_n)_{n\in\mathbb{N}}$ uniformly converges in derivative to f with rate of convergence $\frac{1}{2^n}$ if there exists $K>0$ and $N\in\mathbb{N}$, such that for any $n > N$:

$$\sup_{\substack{i\in\{0,..,2^n-1\}\\ j\in\{0,..,2^n\}}} \left\| 2^n \left[P_n\left(\frac{i+1}{2^n}, \frac{j}{2^n}\right) - P_n\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \right] - \frac{\partial f}{\partial u}\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \right\| \leq \frac{K}{2^n},$$

and

$$\sup_{\substack{i\in\{0,..,2^n\}\\ j\in\{0,..,2^n-1\}}} \left\| 2^n \left[P_n\left(\frac{i}{2^n}, \frac{j+1}{2^n}\right) - P_n\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \right] - \frac{\partial f}{\partial v}\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \right\| \leq \frac{K}{2^n}.$$

Corollary 1 *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of parametrized triangulations $P_n : [0,1]^2 \to \mathbb{R}^3$ and $f : [0,1]^2 \to \mathbb{R}^3$ be a parametrized surface of class $\mathcal{C}^2$, such that:*

- *a) the parameter domain of each P_n is a triangular grid,*
- *b) $(P_n)_{n\in\mathbb{N}}$ uniformly converges to f with rate of convergence $\frac{1}{2^n}$,*
- *c) $(P_n)_{n\in\mathbb{N}}$ uniformly converges in derivative to f with rate of convergence $\frac{1}{2^n}$,*
- *d) f is regular, i.e. $\forall (u,v) \in [0,1]^2$ $\frac{\partial f}{\partial u}(u,v) \wedge \frac{\partial f}{\partial v}(u,v) \neq 0$.*

Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of polygonal curves $C_n : [0,1] \to \mathbb{R}^3$ with uniform parametrization such that C_n is an interior geodesic of P_n. If $(C_n)_{n\in\mathbb{N}}$ converges in the sup norm sense toward a curve C which is interior to S, then C is of class $\mathcal{C}^2$ and is a geodesic of S.

The proof of this corollary is given in Section 6.7.

The previous corollary can be easily applied to some subdivision schemes. As an example, we give the following corollary concerning subdivision scheme for bicubic B-spline (see for example [10] for details on B-splines). First remark that the subdivision scheme for bicubic B-splines generates a sequence of quadrangulations. Each quadrangulation can be considered as a triangulation by dividing each quadrangle into two triangles.

Corollary 2 *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of triangulations (or quadrangulations) defined by the subdivision scheme for bicubic B-spline that is converging to a regular B-spline f (i.e. satisfies assumption d) of Corollary 1).*
Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of polygonal curves $C_n : [0,1] \to \mathbb{R}^3$ with uniform parametrization such that C_n is an interior geodesic of P_n. If $(C_n)_{n\in\mathbb{N}}$ converges in the sup norm sense toward a curve C which is interior to S, then C is of class $\mathcal{C}^2$ and is a geodesic of S.

Proof The assumptions *a)* and *d)* of Corollary 1 are clearly satisfied. Let $u \in [0,1]$. The polygonal curve $v \to P_n(u,v)$ follows the subdivision scheme for cubic B-spline and is uniformly converging to the function $f_u : v \to f(u,v)$. More precisely, there exists $K_u \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ one has (see Theorem 4.12 of [9] or Corollary 3.3 of [8]):

$$\sup_{v\in[0,1]} \|P_n(u,v) - f(u,v)\| \leq \frac{K_u}{2^n}.$$

In fact, one can show that K_u does not depend on u: let us denote by $P_{i,j} = P_0(u_i, v_j)$ the poles of the initial control mesh (where $i, j \in \{0, ..M\}$). The k^{th} pole $P_{u,k}$ of f_u is the evaluation at u of the B-spline whose control net is $P_{0,k}$,...$P_{M,k}$. That implies that $P_{u,k}$ belongs to the convex hull of $P_{0,k}$,...$P_{M,k}$ and then to the convex hull of the $P_{i,j}$.

Since K_u only depends on the maximum distance between two poles of the control net of f_u (see the proof of Theorem 4.11 of [9]), it only depends on the diameter of the convex hull of the $P_{i,j}$ and does not depend on u.

This implies that $(P_n)_{n\in\mathbb{N}}$ uniformly converges toward f. Assumption *b)* is then proved. Similarly, assumption *c)* is also proved on the divided difference scheme (see for example [22] for details on divided difference schemes). □

Remark 3 Corollary 2 directly implies that the result of convergence still holds for Catmull-Clark schemes if the limit curve C does not contain extraordinary points of S (see [3] for details on the Catmull-Clark scheme). By extraordinary point on the limit surface, we precisely mean the limit of the sequence of vertices corresponding to an extraordinary point of the triangulation through successive subdivisions. Indeed, after a sufficient number of iterations, the curve traverses only a finite number of bicubic B-splines patches where Corollary 2 can be applied.

More precisely, the curve C is at a distance greater than $\mu > 0$ from all the extraordinary points. Let us denote by $(P_n)_{n\in\mathbb{N}}$ the sequence of triangulations (or quadrangulations) defined by Catmull-Clark scheme. Then, by compacity, there exists a finite number of triangulations V_n^1,...V_n^p and $0 = t_0 < t_1 < t_2 < ... < t_p = 1$ such that: i) each V_n^i follows a subdivision scheme for bicubic B-spline (as in Corollary 2) and is converging to $S^i \subset S$; ii) for every i, $C_n([t_{i-1}, t_i]) \subset V_n^i$ and $C([t_{i-1}, t_i]) \subset S^i$; iii) $V_n^i \cap V_n^{i+1} \cap C_n([t_{i-1}, t_i])$ is homeomorphic to a connected curve. We then apply Corollary 2 on each V_n^i.

Remark 4 A proof similar to the one of Corollary 2 shows that this result also holds for Bezier surfaces and their successive control nets defined by the De Casteljau algorithm.

6 Proof of Theorem 1

The aim of this section is to prove Theorem 1. The proof being quite long, we first give an overview of each subsection.

6.1 We introduce the definitions needed for the proof, but not usefull for the statement of Theorem 1.

6.2 We give basic lemmas about angles and orthogonal projections onto planes. These lemmas are used in the rest of the proof.

6.3 We give results concerning geodesics on triangulations. We first bound from above the angle of deviation $\beta_{dev}^{3D}(p_n)$ of the geodesic at a vertex p_n. Intuitively, this means that if the triangulation is "almost planar", then the geodesic is "not turning too much". We then also bound from above the angle of deviation $\beta_{dev}^{TS}(p_n)$ of the projection of the geodesic onto a plane. Intuitively, if the plane is "almost parallel" to the triangles of the triangulation, then the projection of the geodesic is "turning much less" than the geodesic itself.

6.4 We bound the number of intersections of a polygonal curve C_n of T_n with the edges of a triangulation T_n. Intuitively, we show that if the curve is "not too long"and if it does "not turn too much", then its number of intersection is "small".

6.5 There is no underlying triangulation in this section. The result gives a sufficient condition on a sequence of curves so that its limit curve is of class $\mathcal{C}^{1,1}$.

6.6 We give here the core of the proof of Theorem 1, that is using the previous sections.

6.7 We prove in this section Corollary 1.

6.1 Preliminary definitions

Let ϵ be smaller than the reach of S. Let T be a triangulation such that ξ induces an injection from T to S. Let denote by $R(T)$ the set of polygonal curves C of T that are linear on each triangle of T, to be more precise, if τ is a triangle (a *triangle* is defined here as a closed simplex, i.e. containing its boundary edges and vertices) of T, the image of each connected component of $\{t \in [0,1]\,, C(t) \in \tau\}$ is a line segment: geodesics on T trivially satisfy this condition. Notice that this condition allows the curve to visit more than once a given triangle τ but, in this case, has to visit the interior of other triangle between two successive visits of τ.

Let $C \in R(T)$ be a polygonal curve that belongs to the tubular neighborhood $\mathcal{V}_\epsilon(S)$ of radius ϵ of S. In the following, if $m \in S$, we denote by P_m^S the orthogonal projection onto the tangent plane of S at the point m.

• The *total curvature of* C is given by:

$$TC_{3D}(C) = \sum_{p\ vertex\ of\ C} \beta_{dev}^{3D}(p),$$

where $\beta_{dev}^{3D}(p)$ is the deviation angle of C at the vertex p (see Figure 3).

Similarly, for $0 \leq t_a < t_b \leq 1$ we denote by $TC_{3D}(C, t_a, t_b)$ the total curvature of the curve C restricted to $[t_a, t_b]$.

• The *tangent total curvature of* C *with respect to* S is defined by

$$TC_{Tan}^S(C) = \sum_{p\ vertex\ of\ C} \beta_{dev}^{TS}(p),$$

where $\beta_{dev}^{TS}(p)$ is the deviation angle of $P_{\xi(p)}^S(C)$ at the vertex $\xi(p)$ (see Figure 3).

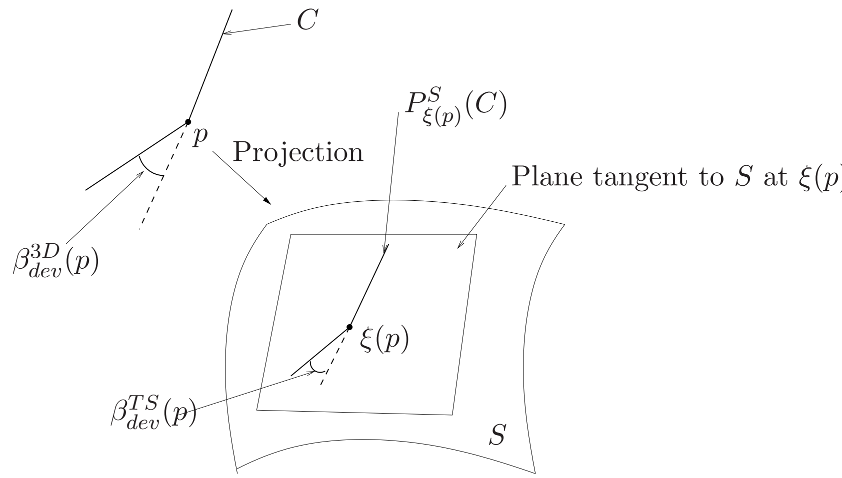

Figure 3: Deviation angle of the curve and of its projection

Similarly, for $0 \leq t_a < t_b \leq 1$ we denote by $TC^S_{Tan}(C, t_a, t_b)$ the tangent total curvature of the curve C restricted to $[t_a, t_b]$.
• Let $\sharp C$ be the number of intersections between C and the edges of T. More precisely, if one denotes by E the set of edges of T and $N_{CC}(X)$ the number of connected components of a set X:

$$\sharp C = \sum_{e \in E} N_{CC}(C([0,1]) \cap e)$$

Notice that each time C transversally crosses an edge away from a vertex, $\sharp C$ is increased of 1 and each time C crosses a vertex "generically" (that is without following an edge), $\sharp C$ is increased of the vertex valency. If it follows an edge from one vertex to the other, it crosses two vertices but the edge is counted once.
Similarly, for $0 \leq t_a < t_b \leq 1$ we denote by $\sharp(C, t_a, t_b)$ the number of intersections between the curve C restricted to $[t_a, t_b]$ and the edges of T.

6.2 Basic lemmas about planes in $\mathbb{R}^3$

In this section, we prove several very usefull basic lemmas.

Lemma 1 *There exists $K > 0$ such that for every planes Π and Π_1 and for every vectors u and v of Π, one has:*

$$|\angle(u, v) - \angle(P_1(u), P_1(v))| \leq K\ \angle(u, v)\ \alpha^2,$$

and also

$$|\angle(u, v) - \angle(P_1(u), P_1(v))| \leq K\ \angle(P_1(u), P_1(v))\ \alpha^2,$$

where P_1 is the orthogonal projection onto Π_1, and α is the angle between Π and Π_1.

Proof

• We put $\theta = \angle(u, v)$ and $\theta_1 = \angle(P_1(u), P_1(v))$. We clearly have

$$\|P_1(u)\| \le \|u\| \le \frac{1}{\cos\alpha}\|P_1(u)\|.$$

The same inequality holds with v. Furthermore a simple calculus gives

$$Area(\Delta) = \frac{1}{\cos\alpha}\ Area(P_1(\Delta)).$$

Now by using the fact that

$$\sin\theta = \frac{Area(\Delta)}{2\ \|u\|\ \|v\|} \quad \text{and} \quad \sin\theta_1 = \frac{Area(P_1(\Delta))}{2\ \|P_1(u)\|\ \|P_1(v)\|},$$

we have

$$\cos\alpha \le \frac{\sin\theta_1}{\sin\theta} \le \frac{1}{\cos\alpha},$$

and then

$$|\sin\theta - \sin\theta_1| = O(\alpha^2)\ \sin\theta = O(\alpha^2)\ \theta. \tag{3}$$

• We put $X = \frac{u}{\|u\|}$ and $Y = \frac{v}{\|v\|}$. Then $\|X - P_1(X)\| \le \sin\alpha \le \alpha$. Furthermore we have

$$\frac{\pi}{2} - \alpha \le \angle\,(P_1(X) - X, Y) \le \frac{\pi}{2} + \alpha,$$

which implies that $| < X - P_1(X), Y > | \le \alpha^2$. Similarly, we also have $| < Y - P_1(Y), X > | \le \alpha^2$ which implies that

$$\begin{aligned} \|P_1(X)\|\ \|P_1(Y)\|\ \cos\theta_1 &= < P_1(X), P_1(Y) > \\ &= < X, Y > + O(\alpha^2) \\ &= \cos\theta + O(\alpha^2). \end{aligned}$$

We then have:

$$|\cos\theta - \cos\theta_1| = O(\alpha^2). \tag{4}$$

• Suppose now that $\theta \in \left[0, \frac{\pi}{4}\right]$. If α is small enough, then Equation (3) implies that $\theta_p \in \left[0, \frac{3\pi}{8}\right]$ and that

$$|\theta_1 - \theta| \le \left(\sup_{x \in \left[0, \sin\frac{3\pi}{8}\right]} |\arcsin'(x)| \right)\ |\sin\theta_1 - \sin\theta| = O(\alpha^2)\ \theta.$$

Remark that the same results holds if $\theta \in \left[\frac{3\pi}{4}, \frac{\pi}{2}\right]$.

• Suppose now that $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. If α is small enough, then Equation (4) implies that

$$|\theta_1 - \theta| \le \left(\sup_{x \in \left[0, \cos\frac{\pi}{8}\right]} |\arccos'(x)| \right)\ |\cos\theta - \cos\theta_1| = O(\alpha^2) = O(\alpha^2)\ \theta.$$

The second inequation of Lemma 1 is a direct consequence of the first one (with a larger constant). $\square$

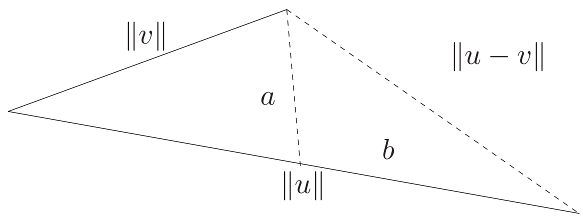

Figure 4: Proof of Lemma 2

Lemma 2 *Let u and v be two vectors and $\epsilon > 0$ such that $|\|u\| - 1| \le \epsilon$ and $|\|v\| - 1| \le \epsilon$. We then have:*

$$2 \ \sin \frac{\angle(u,v)}{2} \le \frac{\|u-v\|}{\min(\|u\|, \|v\|)} \le 2 \ \sin \frac{\angle(u,v)}{2} + \frac{2\epsilon}{1-\epsilon}.$$

Proof Suppose that $\|v\| \le \|u\|$. We then have

$$\frac{\|u-v\|}{\|v\|} \ge \frac{a}{\|v\|} = 2 \ \sin \frac{\angle(u,v)}{2}.$$

We also have

$$\frac{\|u-v\|}{\|v\|} \le \frac{a+b}{\|v\|} = 2 \ \sin \frac{\angle(u,v)}{2} + \frac{\|u\| - \|v\|}{\|v\|} \le 2 \ \sin \frac{\angle(u,v)}{2} + \frac{2\epsilon}{1-\epsilon}.$$

$\square$

Lemma 3 *For some constant K, if P_1 and P_2 are the respective projections on two planes Π_1 and Π_2 with $\angle(\Pi_1, \Pi_2) = \theta \le \frac{1}{10}$ and if u and v are two unit vectors and $\gamma > 0$ a number such that:*

- $\angle(\Pi_2, u) \le \frac{1}{10}$
- $\angle(\Pi_2, v) \le \frac{1}{10}$
- $\angle(\Pi_1, u) \le \gamma \le \frac{1}{10}$
- $\angle(\Pi_1, v) \le \gamma \le \frac{1}{10}$,

one has:

$$\angle(P_2(u), P_2(v)) \le K \left[\angle(P_1(u)), P_1(v)) + \sin\theta \, \angle(u,v) + \gamma^2\right].$$

Proof Let $\delta = v - u$ The affine projection P_1 induces a corresponding projection between vectors which is also denoted P_1. δ can be splited in $\delta = P_1(\delta) + \widetilde{\delta}$ with $P_1(\delta)$ and $\widetilde{\delta}$ respectively parallel and orthogonal to Π_1, which entails $P_1(\delta) = P_1(\delta_1)$ and $\|\widetilde{\delta}\| \le \|\delta\|$. One has $P_2(\delta) = P_2(P_1(\delta)) + P_2(\widetilde{\delta})$ and:

$$\begin{aligned} \|P_2(\delta)\| &\le \|P_2(P_1(\delta))\| + \|P_2(\widetilde{\delta})\| \\ &\le \|P_1(\delta)\| + \|P_2(\widetilde{\delta})\| \\ &\le \|P_1(\delta)\| + \sin\theta \|\widetilde{\delta}\| \\ &\le \|P_1(\delta)\| + \sin\theta \|\delta\|. \end{aligned}$$

From $\angle(\Pi_2, u) \leq \frac{1}{10}$ and $\angle(\Pi_2, v) \leq \frac{1}{10}$, one has:

$$\min(\|P_2(u)\|, \|P_2(v)\|) \geq \cos\frac{1}{10}.$$

Using twice Lemma 2 we have:

$$2\sin\frac{\angle(P_2(u), P_2(v))}{2} \leq \frac{1}{\cos\frac{1}{10}}\|P_2(\delta)\|$$

and

$$\|P_1(\delta)\| \leq 2\sin\frac{\angle(P_1(u), P_1(v))}{2} + O(\gamma^2).$$

This gives:

$$2\sin\frac{\angle(P_2(u), P_2(v))}{2} \leq \frac{1}{\cos\frac{1}{10}}\left[2\sin\frac{\angle(P_1(u), P_1(v))}{2} + O(\gamma^2) + \sin\theta\|\delta\|\right].$$

Using that $\|\delta\| = 2\ \sin\frac{\angle(u,v)}{2}$ and that, for any angle $\beta \in [0, \pi]$ one has $\frac{2}{\pi}\beta \leq 2\frac{\sin\beta}{2} \leq \beta$ we get the above bound on $\angle(P_2(u), P_2(v))$. $\square$

Lemma 4 *There exists $K > 0$ such that for every planes Π_1 and Π_2 and for every unit vectors u and v of $\mathbb{R}^3$ such that $\angle(u, \Pi_i) \leq \frac{\pi}{4}$ and $\angle(v, \Pi_i) \leq \frac{\pi}{4}$, we have:*

$$|\angle(P_1(u), P_1(v)) \ - \ \angle(P_2(u), P_2(v))| \leq K\ \gamma,$$

where P_1 and P_2 denote the respective projections on Π_1 and Π_2, and γ denotes the angle between Π_1 and Π_2.

Proof We put $P = P_1 - P_2$, $\theta = \angle(u, v)$, $\theta_1 = \angle(P_1(u), P_1(v))$ and $\theta_2 = \angle(P_2(u), P_2(v))$.
Case 1: $\theta \in \left[\frac{\pi}{10}, \pi - \frac{\pi}{10}\right]$
Let X be a vector of $\mathbb{R}^3$. We put $X_1 = P_1(X)$ and $X_1^\perp = X - X_1$. We then have

$$\|P(X)\| = \|P(X_1) + P(X_1^\perp)\| \leq \|P(X_1)\| + \|P(X_1^\perp)\|.$$

On the other hand

$$\|P(X_1)\| = \|X_1 - P_2(X_1)\| \leq \sin\gamma\ \|X_1\| \leq \gamma\ \|X\|,$$

and

$$\|P(X_1^\perp)\| = \|P_2(X_1^\perp)\| \leq \sin\gamma\ \|X_1^\perp\| \leq \gamma\ \|X\|,$$

which implies that $\|P(X)\| \leq 2\ \gamma\ \|X\|$. We then have:

$$|\ \|P_1(u)\| \ - \ \|P_2(u)\|\ | \leq \|P(u)\| \leq 2\,\gamma \quad \text{and} \quad |\,\|P_1(v)\| - \|P_2(v)\| \leq 2\,\gamma.$$

Furthermore, the fact that $\|P_i(u)\| \geq \frac{1}{\sqrt{2}}$, $\|P_i(v)\| \geq \frac{1}{\sqrt{2}}$ and

$$\cos\theta_i = \frac{\|P_i(u)\|^2 + \|P_i(v)\|^2 - \|P_i(u - v)\|^2}{2\ \|P_i(u)\|\ \|P_i(v)\|},$$

implies that there exists $K > 0$ such that

$$|\cos\theta_1 - \cos\theta_2| \leq K\ \gamma.$$

Then there exists $K > 0$ such that $|\theta_1 - \theta_2| \leq K\ \gamma$.
Case 2: $\theta \in \left[0, \frac{\pi}{10}\right]$
We denote by Δ a triangle with edges u, v and $u - v$ and by θ' and θ'' the two other angles of Δ. Remark that $\theta' = \theta'' \in \left[\frac{\pi}{2} - \frac{\pi}{20}, \frac{\pi}{2}\right]$. We also denote by θ'_i and θ''_i the two angles of the triangle $P_i(\Delta)$. We then have $|\theta'_1 - \theta'_2| \leq K\ \gamma$ and $|\theta''_1 - \theta''_2| \leq K\ \gamma$. Then

$$|\theta_1 - \theta_2| = |(\pi - \theta'_1 - \theta''_1) - (\pi - \theta'_2 - \theta''_2)| \leq 2\ K\ \gamma.$$

Case 3: $\theta \in \left[\pi - \frac{\pi}{10}, \pi\right]$
Since $\pi - \theta \in \left[0, \frac{\pi}{10}\right]$, we have

$$|\theta_1 - \theta_2| = |(\pi - \theta_1) - (\pi - \theta_2)| \leq K\ \gamma.$$

□

6.3 Majoration of the deviation angles of a geodesic

Proposition 3 *There exists K_1, such that for every n: if C_n is a geodesic of T_n and p_n is a vertex of C_n, then we have:*

$$\beta^{3D}_{dev}(p_n) \leq K_1\ \alpha_n, \quad \text{and} \quad \beta^{TS}_{dev}(p_n) \leq K_1\ \alpha_n^2,$$

where α_n is the maximal angle between all the triangles of T_n containing p_n and $\Pi^S_{\xi(p_n)}$.

Proof

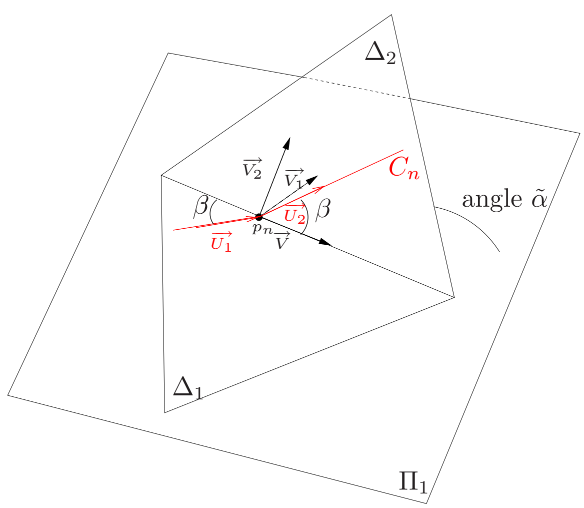

Figure 5: Proof of Proposition 3 - Case 1

Case 1: p_n is not a vertex of T_n
We denote by Δ_1 and Δ_2 the two triangles containing p_n and by e

their common edge. We denote by Π_1 and Π_2 the planes containing respectively Δ_1 and Δ_2. We consider the following unit vectors: $\overrightarrow{V}$ is colinear to e; $\overrightarrow{U_1}$ and $\overrightarrow{U_2}$ are colinear to C_n respectively in the planes Π_1 and Π_2 and oriented with the orientation of curve C_n; $\overrightarrow{V_1}$ is the vector in the plane Π_1 orthogonal to $\overrightarrow{V}$; $\overrightarrow{V_2}$ is the vector in the plane Π_2 orthogonal to $\overrightarrow{V}$ and $\overrightarrow{n_1}$ is the vector normal to P_1. We denote by $\widetilde{\alpha}$ the angle between Δ_1 and Δ_2, and β is the incident and refracted angle of C_n at p_n (see Figure 5). One has:

$$\overrightarrow{V_2} = \cos\widetilde{\alpha}\overrightarrow{V_1} + \sin\widetilde{\alpha}\overrightarrow{n_1}$$

and

$$\begin{aligned}\overrightarrow{U_1} &= \cos\beta\overrightarrow{V} + \sin\beta\overrightarrow{V_1}\\ \overrightarrow{U_2} &= \cos\beta\overrightarrow{V} + \sin\beta\overrightarrow{V_2}.\end{aligned}$$

Which gives:

$$\overrightarrow{U_2} - \overrightarrow{U_1} = \sin\beta\left((\cos\widetilde{\alpha} - 1)\overrightarrow{V_1} + \sin\widetilde{\alpha}\overrightarrow{n_1}\right)$$

then

$$2\ \sin\left(\frac{\beta^{3D}_{dev}(p)}{2}\right) = \|\overrightarrow{U_2} - \overrightarrow{U_1}\| \leq \left(\frac{1}{2}\widetilde{\alpha}^2 + \widetilde{\alpha}\right) \leq 2\widetilde{\alpha} \leq 2\alpha_n. \qquad (5)$$

Then, there exists K_1 such that

$$\beta^{3D}_{dev}(p) \leq K_1\ \alpha_n.$$

Similarly, one has, $P^S_{\xi(p_n)}$ being linear:

$$P^S_{\xi(p_n)}\left(\overrightarrow{U_2}\right) - P^S_{\xi(p_n)}\left(\overrightarrow{U_1}\right) = \sin\beta\left((1-\cos\widetilde{\alpha})P^S_{\xi(p_n)}\left(\overrightarrow{V_1}\right) + \sin\widetilde{\alpha}P^S_{\xi(p_n)}\left(\overrightarrow{n_1}\right)\right).$$

That gives :

$$\left\|P^S_{\xi(p_n)}\left(\overrightarrow{U_2}\right) - P^S_{\xi(p_n)}\left(\overrightarrow{U_1}\right)\right\| \leq \sin\beta\left((1-\cos\widetilde{\alpha})\left\|P^S_{\xi(p_n)}\left(\overrightarrow{V_1}\right)\right\| + \sin\widetilde{\alpha}\left\|P^S_{\xi(p_n)}\left(\overrightarrow{n_1}\right)\right\|\right)$$

We know that:

$$\left\|P^S_{\xi(p_n)}\left(\overrightarrow{n_1}\right)\right\| \leq \sin\alpha_n \leq \alpha_n.$$

Which gives:

$$\left\|P^S_{\xi(p_n)}\left(\overrightarrow{U_2}\right) - P^S_{\xi(p_n)}\left(\overrightarrow{U_1}\right)\right\| \leq \left(\frac{1}{2}\widetilde{\alpha}^2 + \widetilde{\alpha}\ \alpha_n\right) \leq 2{\alpha_n}^2. \qquad (6)$$

Lemma 2 then implies:

$$\beta^{TS}_{dev}(p_n) \leq K_1\ \alpha_n^2.$$

<u>Case 2:</u> p_n is a vertex of T_n

• In this case, the sum of all the angles of T_n at p_n is necessarily greater

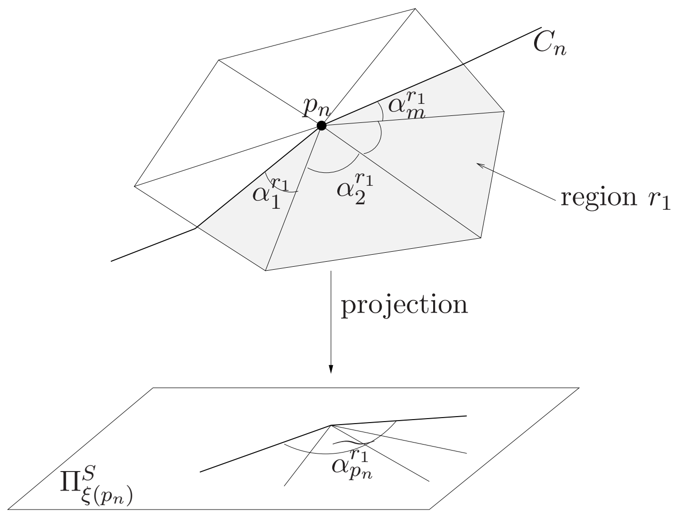

Figure 6: Proof of Proposition 3 - Case 2

than 2π (see Section 2.4). To be more precise, the curve C_n separates the set of the triangles of T_n containing p_n into two connected regions r_1 and r_2 (see Figure 6). If one denotes by $\alpha_p^{r_1}$ the sum of the angles $\alpha_i^{r_1}$ of the triangles of region r_1 at p_n (*resp.* by $\alpha_p^{r_2}$ the angles $\alpha_i^{r_2}$ of the triangles of region r_2 at p_n), since C_n is a geodesic, one has:

$$\alpha_{p_n}^{r_1} \geq \pi \quad \text{and} \quad \alpha_{p_n}^{r_2} \geq \pi.$$

By Lemma 1, the angular defect $|2\pi - (\alpha_p^{r_1} + \alpha_p^{r_2})|$ is less than $2K\pi\alpha_n^2$. We then have $\pi \leq \alpha_p^{r_1} \leq \pi + 2K\pi\alpha_n^2$. Let denote by $\widetilde{\alpha_i^{r_1}}$ the angle of the projection onto $\Pi^S_{\xi(p_n)}$ of $\alpha_i^{r_1}$. We denote by $\widetilde{\alpha_p^{r_1}}$ the sum of the $\widetilde{\alpha_i^{r_1}}$. By using again Lemma 1, we have $\left|\alpha_p^{r_1} - \widetilde{\alpha_p^{r_1}}\right| \leq 2K\pi\alpha_n^2$ and then

$$\beta_{dev}^{TS}(p_n) = \left|\widetilde{\alpha_p^{r_1}} - \pi\right| \leq \left|\widetilde{\alpha_p^{r_1}} - \alpha_p^{r_1}\right| + \left|\alpha_p^{r_1} - \pi\right| \leq 4K\pi\alpha_n^2.$$

• The curve C_n is included in the neighborhood of p_n in two triangles Δ_1 and Δ_2. We denote by Π_1 the plane containing Δ_1 and by P_1 the orthogonal projection onto Π_1. Let $\overrightarrow{U_1}$ and $\overrightarrow{U_2}$ denote respectively the two unitary vectors colinear to C_n in Δ_1 and in Δ_2, oriented with the orientation of curve C_n. We put $\overrightarrow{U_2}^{\perp} = P_1(\overrightarrow{U_2}) - \overrightarrow{U_2}$. Similarly as before, by using Lemma 1 and the projection P_1 onto Π_1, we have

$$\angle(P_1(\overrightarrow{U_1}), P_1(\overrightarrow{U_2})) \leq 4K\pi\alpha_n^2.$$

We also have:

$$\overrightarrow{U_1} - \overrightarrow{U_2} = \overrightarrow{U_1} - P_1(\overrightarrow{U_2}) + \overrightarrow{U_2}^{\perp} = P_1(\overrightarrow{U_1}) - P_1(\overrightarrow{U_2}) + \overrightarrow{U_2}^{\perp}.$$

Since $\|P_1(\overrightarrow{U_1})\| = 1$ and $\|P_1(\overrightarrow{U_2})\| \geq \cos\alpha_n$, by Lemma 2 we have:

$$\begin{array}{rcl} 2\sin\frac{\angle(\overrightarrow{U_1},\overrightarrow{U_2})}{2} & = & \|\overrightarrow{U_1}-\overrightarrow{U_2}\| \\ & \leq & \|P_1(\overrightarrow{U_1})-P_1(\overrightarrow{U_2})\| + \|\overrightarrow{U_2}^{\perp}\| \\ & \leq & 2\ \sin\frac{\angle(P_1(\overrightarrow{U_1}),P_1(\overrightarrow{U_2}))}{2} + O(\alpha_n{}^2) + \sin(\alpha_n) \\ & \leq & \angle(P_1(\overrightarrow{U_1}),P_1(\overrightarrow{U_2})) + O(\alpha_n) \\ & = & O(\alpha_n). \end{array}$$

This implies that there exists K_1 such that

$$\beta^{3D}_{dev}(p_n) = \angle(\overrightarrow{U_1},\overrightarrow{U_2}) \leq K_1\ \alpha_n.$$

□

6.4 Majoration of $\sharp C_n$

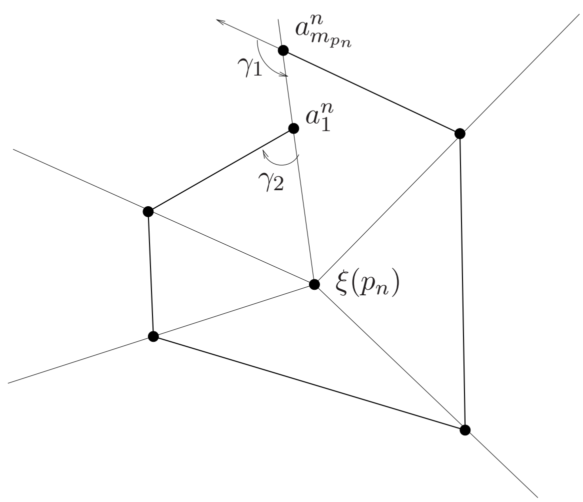

Figure 7: Projection of C_n onto the plane $\Pi^S_{\xi(p_n)}$ (Proposition 4 - case 1)

Proposition 4 *There exists a constant K_2, such that for any curve $C_n \in R(T_n)$, one has:*

$$\sharp(C_n) \leq K_2\left[1+\ TC^{TS}_{Tan}(C_n) + 2^n\ l(C_n)\right].$$

Proof **of Proposition 4**
If p_n is a vertex of T_n, let us denote by $Cell(p_n)$ the set of points of the triangulation T_n that are closer to p_n than to the other vertices of T_n. Let η_{min} denote the length of the smallest edge of T_n, and by θ_{min} the smallest angle of the triangulation T_n. First remark that the smallest altitude is larger than $\eta_{min}\sin(\theta_{min})$. We put $l_{min} = \frac{\eta_{min}\sin(\theta_{min})}{4}$.
• Let us first consider a curve $C_n \in \mathcal{R}(T_n)$ that satisfies:

$$l(C_n) \leq l_{min} \quad \text{and} \quad TC^{TS}_{Tan}(C_n) \leq \pi.$$

We are going to show that the number $\sharp C_n$ of intersections between C_n and the edges of T_n is bounded by a constant independant of n.
<u>Case 1:</u> There exists a vertex p_n of T_n such that the distance from p_n to C_n is less than l_{min}.
Since the length of the curve C_n is less than l_{min}, every point m of C_n is a distance less than $2l_{min}$ from the vertex p_n. This implies that $C_n \subset Cell(p_n)$.

By definition, the curve C_n of $\mathcal{R}(T_n)$ "crosses" every intersected edge. This implies that C_n either contains the vertex p_n or is turning around the vertex p_n without changing the sense (in the clockwise sense or in the counter-clockwise sense).
− If $p_n \in C_n$, then C_n follows 0, 1 or 2 edges and only contains the vertex p_n. This implies that $\sharp C_n$ is less than the valence of p_n which is uniformly bounded from above by a constant V (see Remark 1).
− If C_n is turning around p_n, we are going to show that C_n cannot intersect three times the same edge e. If C_n is intersecting twice the same edge e, then the discrete Gauss-Bonnet formulae of the projecton $P^S_{\xi(p_n)}(C_n)$ of the curve C_n onto the plane $\Pi^S_{\xi(p)}$ implies that (see Figure 7)

$$TC_{3D}(P^S_{\xi(p_n)}(C_n)) + \gamma_1 + \gamma_2 \geq 2\pi,$$

where $\gamma_1 \in [0, \pi]$ and $\gamma_2 \in [-\pi, 0]$. We the have $|\gamma_1 + \gamma_2| \leq \pi$ and:

$$TC_{3D}(P^S_{\xi(p_n)}(C_n)) \geq 2\pi - (\gamma_1 + \gamma_2) \geq \pi.$$

Let us denote by a^n_1,...$a^n_{m_{p_n}}$ consecutive vertices of $P^S_{\xi(p_n)}(C_n)$ such that a^n_1 and $a^n_{m_{p_n}}$ belong to the same edge e. Remark that $m_{p_n} - 1$ is equal to the valency of p_n and thus is less than V. Since the angle between $\Pi^S_{\xi(a^n_i)}$ and $\Pi^S_{\xi(p_n)}$ is less than $\frac{1}{2^n}$, Lemma 4 implies that:

$$TC^{TS}_{Tan}(C_n) \geq TC_{3D}(P^S_{\xi(p_n)}(C_n)) - \frac{K\ m_{p_n}}{2^n} \geq \pi - \frac{K\ V}{2^n}.$$

Let us now suppose that C_n is intersecting three times an edge e. Then we have that:

$$TC^{TS}_{Tan}(C_n) \geq 2\pi - \frac{2K\ V}{2^n} > \pi,$$

for n large enough, which contradicts the assumption made on the curve C_n. This implies that the number $\sharp C_n$ of intersections between C_n and T_n is bounded from above by twice the valency of p_n, which is less than $2V$. Then $\sharp C_n$ is uniformly bounded from above.
<u>Case 2:</u> The distance between C_n and all the vertices of T_n is larger than l_{min}.
Let $\Delta_n = p_n q_n r_n$ denote a triangle of T_n that is intersected by C_n. The intersection is a segment $[a_n, b_n]$ and we denote by θ_{p_n} the angle at p_n (see Figure 8). We can suppose that $p_n a_n \leq p_n b_n$ and we have:

$$a_n b_n \geq 2 \sin\left(\frac{\theta_{p_n}}{2}\right) p_n a_n \geq 2 \sin\left(\frac{\theta_{min}}{2}\right) l_{min}.$$

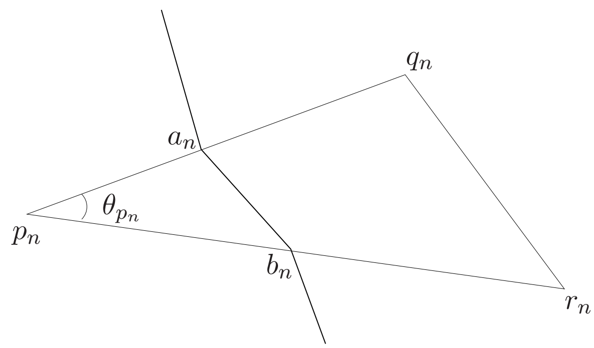

Figure 8: Proof of Proposition 4 - case 2

Then the number $\sharp C_n$ of intersection between C_n and the edges of T_n is less than:

$$1+\frac{l(C_n)}{2\sin\left(\frac{\theta_{min}}{2}\right)l_{min}} \leq 1+\frac{1}{2\sin\left(\frac{\theta_{min}}{2}\right)}.$$

In the two cases, $\sharp C_n$ is bounded by a constant independant on n. That implies that there exists a constant $\widetilde{K}$ such that:

$$\sharp(C_n) \leq \widetilde{K}.$$

• Let us now consider any curve $C_n \in \mathcal{R}(T_n)$. The curve C_n can be subdivided in N curves C_n^1,...,C_n^N that satisfy:

$$l(C_n^i) \leq l_{min} \quad \text{and} \quad TC_{Tan}^{TS}(C_n^i) \leq \pi,$$

where

$$N \leq \frac{l(C_n)}{l_{min}} + \frac{TC_{Tan}^{TS}(C_n)}{\pi} + 1.$$

Then there exists a constant K_2 such that:

$$\sharp(C_n) \leq N\ \widetilde{K} \leq K_2\left[1+\ TC_{Tan}^{TS}(C_n) + 2^n\ l(C_n)\right].$$

□

6.5 A sufficient condition for the regularity of the limit curve

Proposition 5 *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of polygonal curves $C_n : [0,1] \to \mathbb{R}^3$, with uniform parametrization that converges toward a non constant curve C in the sup norm sense. If*

$$\exists k_1, k_2,\ \forall t_a, t_b \in [0,1] \quad TC_{3D}(C_n, t_a, t_b) \leq \frac{k_1}{2^n} + k_2\ l(C_n, t_a, t_b),$$

then the curve C has uniform parametrization, is of class $\mathcal{C}^{1,1}$ and has curvature bounded by k_2. Moreover $l(C) = \lim_{n\to\infty} l(C_n)$ and for any $t_0 \in (0,1)$:

$$\lim_{n\to\infty} \frac{dC_n}{dt^+}(t_0) = \frac{dC}{dt}(t_0).$$

Proof One first proves a few lemmas.

Lemma 5 *In the conditions of Proposition 5, for any $\theta > 0$, there is an integer number N such that for any $n, m \geq N$ and any $t \in [0,1)$:*

$$\angle\left(\frac{dC_n(t)}{dt^+}, \frac{dC_m(t)}{dt^+}\right) < \theta. \tag{7}$$

Proof Recall that, because the curves C_n have uniform parametrization, $\left\|\frac{dC_n}{dt^+}(t)\right\|$ is constant on $[0,1]$. And, as C is non constant and $C_n \to C$ in the sup norm sense, let $t < t'$ be such that $C(t') \neq C(t)$. Then, for some N_0, $n \geq N_0 \Rightarrow \|C - C_n\|_\infty < \frac{1}{4}\|C(t') - C(t)\|$, which entails:

$$\|C_n(t') - C_n(t)\| > \frac{1}{2}\|C(t') - C(t)\|.$$

Therefore, for $\underline{l} = \frac{1}{2}\|C(t') - C(t)\|$, for any $n \geq N_0$, $\left\|\frac{dC_n}{dt^+}(t)\right\| > \underline{l} > 0$. Let $N \geq N_0$ be such that:

$$\frac{k_1}{2^N} < \frac{\theta}{16}, \tag{8}$$

and:

$$\forall n \geq N,\ \|C_n - C\|_\infty < \frac{\theta^2}{128k_2}. \tag{9}$$

Notice that if Lemma 5 holds for $\theta < \min\left(\frac{\pi}{2}, k_2\underline{l}\right)$, it holds in general. We proceed by contradiction. Let us assume that the assertion of the lemma does not hold for some $t \in [0, \frac{1}{2}]$, and $\theta < \min\left(\frac{\pi}{2}, k_2\underline{l}\right)$. More precisely, let us assume that, for some $t \in [0, \frac{1}{2}]$ and $n, m \geq N$, one has:

$$\angle\left(\frac{dC_n(t)}{dt^+}, \frac{dC_m(t)}{dt^+}\right) \geq \theta. \tag{10}$$

Without loss of generality, one can assume that:

$$\left\|\frac{dC_n}{dt^+}(t)\right\| \geq \left\|\frac{dC_m}{dt^+}(t)\right\|. \tag{11}$$

We consider:

$$t' = t + \frac{\theta}{16k_2\left\|\frac{dC_n}{dt^+}(t)\right\|}.$$

Notice that $\theta < k_2\underline{l}$ entails $t' < 1$. One has $l(C_n, t, t') \leq \frac{\theta}{16k_2}$ which gives, with the assumption in Proposition 5 together with inequation (8):

$$TC_{3D}(C_n, t, t') \leq \frac{k_1}{2^n} + k_2\ l(C_n, t, t') \leq \frac{\theta}{8}.$$

Therefore, one has, for any $\tau \in [t, t']$:

$$\left\|\frac{dC_n}{dt^+}(\tau) - \frac{dC_n}{dt^+}(t)\right\| \leq 2\sin\left(\frac{\theta}{16}\right)\left\|\frac{dC_n}{dt^+}\right\|,$$

and:

$$\begin{aligned}
\left\| \int_t^{t'} \frac{dC_n}{dt^+}(\tau)d\tau \; - \; (t'-t)\frac{dC_n}{dt^+}(t) \right\| &= \int_t^{t'} \left(\frac{dC_n}{dt^+}(\tau) - \frac{dC_n}{dt^+}(t) \right) \\
&\leq (t'-t) \left\| \frac{dC_n}{dt^+}(\tau) - \frac{dC_n}{dt^+}(t) \right\| \\
&\leq (t'-t) 2\sin(\frac{\theta}{16}) \left\| \frac{dC_n}{dt^+} \right\| \\
&= 2\sin(\frac{\theta}{16})\frac{\theta}{8k_2}.
\end{aligned}$$

Which gives:

$$\left\| \int_t^{t'} \frac{dC_n}{dt^+}(\tau)d\tau \; - \; (t'-t)\frac{dC_n}{dt^+}(t) \right\| \leq \frac{\theta^2}{64k_2}. \tag{12}$$

And similarly, using inequation (11), one has:

$$\left\| \int_t^{t'} \frac{dC_m}{dt^+}(\tau)d\tau \; - \; (t'-t)\frac{dC_m}{dt^+}(t) \right\| \leq \frac{\theta^2}{64k_2}. \tag{13}$$

On another hand, inequations (11) and (10) entail:

$$\left\| (t'-t)\frac{dC_n}{dt^+}(t) - (t'-t)\frac{dC_m}{dt^+}(t) \right\| \geq \sin\theta \left\| (t'-t)\frac{dC_n}{dt^+}(t) \right\|,$$

which gives:

$$\left\| (t'-t)\frac{dC_n}{dt^+}(t) - (t'-t)\frac{dC_m}{dt^+}(t) \right\| \geq \sin\theta\frac{\theta}{8k_2} > \frac{\theta^2}{16k_2}. \tag{14}$$

But:

$$\begin{aligned}
\left\| \int_t^{t'} \tfrac{dC_n}{dt^+}(\tau)d\tau - \int_t^{t'} \tfrac{dC_m}{dt^+}(\tau)d\tau \right\| &= \left\| C_n(t') - C_n(t) - C_m(t') + C_m(t) \right\| \\
&\leq 2\left\| C_n - C \right\|_\infty,
\end{aligned}$$

which gives, using (9):

$$\left\| \int_t^{t'} \frac{dC_n}{dt^+}(\tau)d\tau - \int_t^{t'} \frac{dC_m}{dt^+}(\tau)d\tau \right\| \leq \frac{\theta^2}{32k_2}.$$

This last equation can not hold together with inequations (14), (12) and (13). Therefore, inequation (10) does not hold and the lemma is proved for $t \in [0, \frac{1}{2}]$. By reparametrizattion of the curves by $t \mapsto 1-t$, one gets the same property for $t \in [\frac{1}{2}, 1]$ but expressed with the left derivatives. However, in the condition of the lemma, the left and right derivatives satisfy:

$$\forall n,\ \forall t,\ \left\| \frac{dC_n}{dt^-}(t) - \frac{dC_n}{dt^+}(t) \right\| \leq \frac{k_1}{2^n}.$$

This entails, using that $\forall n \geq N_0$, $\left\|\frac{dC_n}{dt^+}(t)\right\| > \underline{l} > 0$ that both have same uniform limit angle. □

Lemma 5 gives a Cauchy sequence condition on the angle allows to derive the following lemma on Cauchy sequence conditions on right derivatives.

Lemma 6 *In the conditions of Proposition 5, the sequence of $\frac{dC_n}{dt^+}(t)$ is bounded by some number $\tilde{l}$. Moreover, it is a Cauchy sequence, that is, for any $\epsilon > 0$, there is an integer number N such that for any $n, m \geq N$ and any $t \in [0, 1)$:*

$$\left\|\frac{dC_n(t)}{dt^+} - \frac{dC_m(t)}{dt^+}\right\| < \epsilon. \tag{15}$$

Proof We first claim that, for any $\beta > 0$, there is N such that, for any $n, m \geq N$ and $t \in [0, \frac{1}{2}]$, one has:

$$\left|\left\|\frac{dC_n(t)}{dt^+}\right\| - \left\|\frac{dC_m(t)}{dt^+}\right\|\right| \leq \beta \max\left(\left\|\frac{dC_n(t)}{dt^+}\right\|, \left\|\frac{dC_m(t)}{dt^+}\right\|\right). \tag{16}$$

As in the proof of Lemma 5, we consider N_0 and $\underline{l} > 0$ such that $\forall n \geq N_0$, $\left\|\frac{dC_n}{dt^+}(t)\right\| > \underline{l} > 0$. We consider $\theta < \min\left(\frac{\pi}{2}, k_2\underline{l}\right)$ such that $(1 - \cos\theta) < \frac{\beta}{2}$ and $N_1 \geq N_0$ such that

$$\frac{k_1}{2^{N_1}} < \frac{\theta}{2}. \tag{17}$$

We consider again $N \geq N_1$, using Lemma 5, such that, $\forall n, m \geq N$:

$$\angle\left(\frac{dC_n(t)}{dt^+}, \frac{dC_m(t)}{dt^+}\right) < \theta \tag{18}$$

and

$$\|C_n - C_m\|_\infty \leq \frac{\theta\beta}{8k_2}. \tag{19}$$

We consider some $t \in [0, \frac{1}{2}]$ and $n, m \geq N$, and we will prove that (16) holds. Without loss of generality, one can assume that $\left\|\frac{dC_n(t)}{dt^+}\right\| \geq \left\|\frac{dC_m(t)}{dt^+}\right\|$. As in the proof of Lemma 5, we consider the interval $[t, t']$:

$$t' = t + \frac{\theta}{2k_2\left\|\frac{dC_n}{dt^+}(t)\right\|}, \tag{20}$$

which gives $(t' - t)\left\|\frac{dC_n}{dt^+}(t)\right\| = \frac{\theta}{2k_2}$ and we get from (17) that, $\forall \tau \in [t, t']$:

$$\angle\left(\frac{dC_n}{dt^+}(\tau), \frac{dC_n}{dt^+}(t)\right) \leq \theta. \tag{21}$$

We consider the unitary vector $\mathbf{e} = \frac{\frac{dC_n}{dt^+}(t)}{\left\|\frac{dC_n}{dt^+}(t)\right\|}$. One has, using Inequality (21):

$$\begin{aligned}
< \mathbf{e}, C_n(t') - C_n(t) > &= < \mathbf{e}, \int_t^{t'} \frac{dC_n}{dt^+}(\tau)d\tau > \\
&= \int_t^{t'} < \mathbf{e}, \frac{dC_n}{dt^+}(\tau) > d\tau \\
&\geq \cos\theta \quad (t'-t)\left\|\frac{dC_n}{dt^+}\right\|.
\end{aligned}$$

Similarly, using (18) and (21), one gets:

$$\begin{aligned}
< \mathbf{e}, C_m(t') - C_m(t) > &= < \mathbf{e}, \int_t^{t'} \frac{dC_m}{dt^+}(\tau)d\tau > \\
&= \int_t^{t'} < \mathbf{e}, \frac{dC_m}{dt^+}(\tau) > d\tau \\
&\leq (t'-t)\left\|\frac{dC_m}{dt^+}\right\|.
\end{aligned}$$

Using (19) we get:

$$|< \mathbf{e}, C_n(t') - C_n(t) > - < \mathbf{e}, C_m(t') - C_m(t) >| \leq \frac{\theta\beta}{4k_2},$$

and the three last inequalities sum up in:

$$\cos\theta \quad (t'-t)\left\|\frac{dC_n}{dt^+}\right\| \leq (t'-t)\left\|\frac{dC_m}{dt^+}\right\| + \frac{\theta\beta}{4k_2}.$$

Dividing both terms by $(t'-t) = \frac{\theta}{2k_2\left\|\frac{dC_n}{dt^+}(t)\right\|}$ gives:

$$\cos\theta \quad \left\|\frac{dC_n}{dt^+}(t)\right\| \leq \left\|\frac{dC_m}{dt^+}\right\| + \frac{\beta}{2}\left\|\frac{dC_n}{dt^+}(t)\right\|,$$

which gives:

$$\left\|\frac{dC_n}{dt^+}(t)\right\| - \left\|\frac{dC_m}{dt^+}\right\| \leq (\frac{\beta}{2} + 1 - \cos\theta)\left\|\frac{dC_n}{dt^+}(t)\right\|.$$

The fact that $(1-\cos\theta) < \frac{\beta}{2}$ proves (16).
Property (16) easily proves the lemma. Indeed, taking $\beta = \frac{1}{2}$ gives that, for some N:

$$n \geq N \Rightarrow \left|\left\|\frac{dC_n}{dt^+}(t)\right\| - \left\|\frac{dC_N}{dt^+}(t)\right\|\right| \leq \frac{1}{2}\max\left(\left\|\frac{dC_n}{dt^+}(t)\right\|, \left\|\frac{dC_N}{dt^+}(t)\right\|\right),$$

which entails:

$$n \geq N \Rightarrow \left\|\frac{dC_n}{dt^+}(t)\right\| \leq 2\left\|\frac{dC_N}{dt^+}(t)\right\|.$$

Therefore, the sequence of $\left\|\frac{dC_n}{dt^+}(t)\right\|$ is bounded by some number $\tilde{l}$ and (16) entails that, for any $\beta > 0$, one has:

$$n \geq N \Rightarrow \left| \left\| \frac{dC_n}{dt^+}(t) \right\| - \left\| \frac{dC_N}{dt^+}(t) \right\| \right| \leq \beta \tilde{l}.$$

This implies that $\left\|\frac{dC_n}{dt^+}(t)\right\|$ is a Cauchy sequence. This fact, combined with Lemma 5, implies that $\frac{dC_n}{dt^+}(t)$ is a Cauchy sequence. □

The fact that for every $t \in (0,1)$, one has $l(C_n) = \left\|\frac{dC_n}{dt^+}(t)\right\|$, implies by Lemma 6, that the sequence of lengths $l(C_n)$ is converging to L. For two points a and b, $d(a,b)$ denotes the euclidean distance between a and b. First one proves the following:

Lemma 7 *In the conditions of Proposition 5, for any ϵ with $0 < \epsilon < \frac{1}{10}$, there is η and an integer number N such that if $0 < t_b - t_a \leq \eta$, then for any t, t' such that $t_a \leq t < t' \leq t_b$ and for any $n \geq N$:*

$$\left\| \frac{C_n(t') - C_n(t)}{t' - t} - \frac{C_n(t_b) - C_n(t_a)}{t_b - t_a} \right\| \leq \epsilon \left\| \frac{C_n(t_b) - C_n(t_a)}{t_b - t_a} \right\| \qquad (22)$$

and:

$$l(C_n)(t_b - t_a)(1 - \epsilon^2) \;\leq\; d\left(C_n(t_a), C_n(t_b)\right) \;\leq\; l(C_n)(t_b - t_a). \qquad (23)$$

Proof of Lemma 7
We put $\eta = \frac{\epsilon}{4 k_2 \tilde{l}}$ and N such that $\frac{k_1}{2^N} \leq \frac{\epsilon}{4}$. If $0 < t_b - t_a \leq \eta$, we then have:

$$l(C_n, t_a, t_b) = l(C_n)\ (t_b - t_a) \leq \tilde{l}\ \frac{\epsilon}{4\ k_2\ \tilde{l}} = \frac{\epsilon}{4\ k_2}.$$

We then have

$$TC_{3D}(C_n, t_a, t_b) \leq \frac{k_1}{2^n} + k_2\ l(C_n, t_a, t_b) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

First remark that the right derivative of C_n exists everywhere. In the following, we denote it by $\frac{dC_n}{dt^+}$. Furthermore, since C_n is differentiable almost everywhere, the Lebesgues integral of its derivative is equal to the Lebesgues integral of $\frac{dC_n}{dt^+}$. For any $t_1, t_2 \in [t_a, t_b]$, one has:

$$\angle\left(\frac{dC_n}{dt^+}(t_1), \frac{dC_n}{dt^+}(t_2)\right) \leq \frac{\epsilon}{2}.$$

Then, since C_n has uniform parametrization, one has:

$$\forall t_1 \in [0,1]\ \left\| \frac{dC_n}{dt^+}(t_1) \right\| = l(C_n),$$

and therefore, for any $t_1, t_2 \in [t_a, t_b]$, one has:

$$\left\| \frac{dC_n}{dt^+}(t_2) - \frac{dC_n}{dt^+}(t_1) \right\| \leq 2\ l(C_n)\ \sin\frac{\epsilon}{4}.$$

That implies that for any $t_1 \in [t_a, t_b]$ and t, t' with $t_a \leq t < t' \leq t_b$:

$$\begin{aligned}\left\|\frac{C_n(t')-C_n(t)}{t'-t}-\frac{dC_n}{dt^+}(t_1)\right\| &= \left\|\frac{1}{t'-t}\int_t^{t'}\left(\frac{dC_n}{dt^+}(\tau)-\frac{dC_n}{dt^+}(t_1)\right)d\tau\right\| \\ &\leq 2l(C_n)\sin\frac{\epsilon}{4}.\end{aligned}$$

Again:

$$\begin{aligned}\left\|\frac{C_n(t')-C_n(t)}{t'-t}-\frac{C_n(t_b)-C_n(t_a)}{t_b-t_a}\right\| &= \left\|\frac{1}{t_b-t_a}\int_{t_a}^{t_b}\left(\frac{C_n(t')-C_n(t)}{t'-t}-\frac{dC_n}{dt^+}(\tau)\right)d\tau\right\| \\ &\leq \frac{1}{t_b-t_a}\int_{t_a}^{t_b}\left\|\frac{C_n(t')-C_n(t)}{t'-t}-\frac{dC_n}{dt^+}(t_1)\right\|d\tau \\ &\leq 2l(C_n)\sin\frac{\epsilon}{4}.\end{aligned} \tag{24}$$

On the other hand:

$$\begin{aligned}\left\|\frac{C_n(t_b)-C_n(t_a)}{t_b-t_a}\right\|^2 &= \left(\frac{1}{t_b-t_a}\int_{t_a}^{t_b}\frac{dC_n}{dt^+}(\tau)d\tau\right)\cdot\left(\frac{1}{t_b-t_a}\int_{t_a}^{t_b}\frac{dC_n}{dt^+}(\tau)d\tau\right) \\ &= \frac{1}{(t_b-t_a)^2}\int_{t_a}^{t_b}\int_{t_a}^{t_b}\frac{dC_n}{dt^+}(\tau_1)\cdot\frac{dC_n}{dt^+}(\tau_2)d\tau_2 d\tau_1.\end{aligned}$$

This gives

$$l(C_n)^2\cos\frac{\epsilon}{2}\leq\left\|\frac{C_n(t_b)-C_n(t_a)}{t_b-t_a}\right\|^2\leq l(C_n)^2$$

and

$$l(C_n)\sqrt{\cos\frac{\epsilon}{2}}\leq\left\|\frac{C_n(t_b)-C_n(t_a)}{t_b-t_a}\right\|\leq l(C_n). \tag{25}$$

Using $\epsilon<\frac{1}{10}$, Equations (24) and (25) imply Equation (22); Equation (25) proves Equation (23). □

We recall that $L=\limsup_{n\to\infty} l(C_n)$. Now, by using the fact that $\|C_n-C\|_\infty\to 0$, Lemma 7 gives that, for any ϵ with $0<\epsilon<\frac{1}{10}$, there is η such that, if $0<t_b-t_a\leq\eta$:

$$\left\|\frac{C(t')-C(t)}{t'-t}-\frac{C(t_b)-C(t_a)}{t_b-t_a}\right\|\leq\epsilon\left\|\frac{C(t_b)-C(t_a)}{t_b-t_a}\right\|.$$

Remark that:

$$\left\|\frac{C_n(t_b)-C_n(t_a)}{t_b-t_a}\right\|\leq l(C_n).$$

Which entails, by taking the limit of both sides as $n\to\infty$:

$$\left\|\frac{C(t_b)-C(t_a)}{t_b-t_a}\right\|\leq L.$$

We have then:

$$\left\|\frac{C(t')-C(t)}{t'-t}-\frac{C(t_b)-C(t_a)}{t_b-t_a}\right\|\leq\epsilon L. \tag{26}$$

For a given real number $t_0 \in [0,1]$, and an integer number $j \geq 1$ we introduce the closed subset $K_j(t_0)$ as:

$$K_j(t_0) = \text{Closure}\left[\left\{\frac{C(t') - C(t)}{t' - t} \mid \max\left(0, t_0 - \frac{1}{j}\right) \leq t < t' \leq \min\left(1, t_0 + \frac{1}{j}\right)\right\}\right].$$

From Equation (26) $K_j(t_0)$ is bounded for j large enough, and therefore compact. It is obviously not empty and decreasing for the inclusion: $K_j(t_0) \supset K_{j+1}(t_0)$. From compactness, the set $K(t_0) = \cap_{j\geq 1} K_j(t_0)$ is not empty and Equation (26) entails that it must be contained in a ball of radius ϵL for arbitrarily small ϵ which entail that $K(t_0)$ is a single point:

$$K(t_0) = \{\psi(t_0)\}.$$

Again, Equation (26) entails:

$$\forall \epsilon > 0, \ \exists h > 0,$$
$$\max(0, t_0 - h) \leq t < t' \leq \min(1, t_0 + h) \Rightarrow \left\|\frac{C(t') - C(t)}{t' - t} - \psi(t_0)\right\| < \epsilon.$$

Therefore, for $0 < t_0 < 1$, $\psi(t_0)$ is the derivative of C at t_0. In fact, the expression above is stronger: it states that C is strictly differentiable (see [4] page 30), which entails that the derivative function is continuous. For $t_0 = 0$ (resp. $t_0 = 1$) this shows that C has a right (resp. left) derivative at 0 (resp. 1). We have proven so far that C is of class $\mathcal{C}^1$.

Lemma 8 *The sequence of right derivatives $\frac{dC_n}{dt^+}$ uniformly converges to $\frac{dC}{dt}$. In other words, for every $\epsilon >$, there exists $N \in \mathbb{N}$ such that:*

$$n > N \ \Rightarrow \ \forall t \in (0,1) \ \left\|\frac{dC_n}{dt^+}(t) - \frac{dC}{dt}(t)\right\| \leq \epsilon.$$

Proof Proof of Lemma 8
We know by Lemma 6 that $\frac{dC_n}{dt^+}$ uniformly converges. We only have to show that $\frac{dC_n}{dt^+}(t_0)$ converges to $\frac{dC}{dt}(t_0)$ for any $t_0 \in (0,1)$. From Lemma 7, for any ϵ such that $0 < \epsilon < \frac{1}{10}$, there is $h_0 > 0$ and an integer number N such that, for $n \geq N$ and $h \leq h_0$, one has:

$$\left\|\frac{C_n(t_0 + h) - C_n(t_0 - h)}{2h} - \frac{dC_n}{dt^+}(t_0)\right\| \leq \epsilon.$$

Since C is of class $\mathcal{C}^1$, we can take h small enough to have:

$$\left\|\frac{C(t_0 + h) - C(t_0 - h)}{2h} - \frac{dC}{dt}(t_0)\right\| \leq \epsilon$$

and let $\widetilde{N} \geq N$ be such that $n \geq \widetilde{N} \Rightarrow \|C_n - C\|_\infty \leq \epsilon h$. One gets:

$$\left\|\frac{C_n(t_0 + h) - C_n(t_0 - h)}{2h} - \frac{C(t_0 + h) - C(t_0 - h)}{2h}\right\| \leq \frac{2\epsilon h}{2h} = \epsilon,$$

which gives, for any $n \geq \widetilde{N}$:

$$\left\| \frac{dC_n}{dt^+}(t_0) - \frac{dC}{dt}(t_0) \right\| \leq 3\epsilon.$$

This is true for arbitrary small ϵ which proves the claim. $\square$

Lemma 8 allows to apply Lebesgues' dominated convergence theorem which entails that, for $0 \leq t_a < t_b \leq 1$:

$$\lim_{n\to\infty} l(C_n, t_a, t_b) = l(C, t_a, t_b), \tag{27}$$

and in particular:

$$\lim_{n\to\infty} l(C_n) = L = l(C).$$

The assumptions of Proposition 5 entail that:

$$\forall t_a, t_b \in [0,1] \quad \angle\left(\frac{dC_n}{dt^+}(t_a),\ \frac{dC_n}{dt^+}(t_b)\right) \leq \frac{k_1}{2^n} + k_2\ l(C_n, t_a, t_b).$$

This, together with Lemma 8 and Equation (27) entail:

$$\forall t_a, t_b \in [0,1] \quad \angle\left(\frac{dC}{dt}(t_a),\ \frac{dC}{dt}(t_b)\right) \leq k_2\ l(C, t_a, t_b),$$

which proves that k_2 bounds the curvature of C. Moreover, Lemma 8 entails that, for any t_0 with $0 < t_0 < 1$:

$$\left\| \frac{dC}{dt}(t_0) \right\| = L = l(C).$$

It follows that C has uniform parametrization and $\frac{dC}{dt}$ is $K-$Lipschitz with $K = L^2 k_2$. $\square$

6.6 Proof of Theorem 1

We take the notations of Theorem 1. Let α_n be the smallest real number such that for every $m \in T_n$, the angle between any triangle Δ containing m and the tangent plane $\Pi^S_{\xi(m)}$ of S at $\xi(m)$ is smaller than α_n. By assumption, we have

$$\alpha_n \leq \frac{K}{2^n}.$$

Step 1: Let us consider a given interval $[t_a, t_b] \subset [0,1]$. By definition, we have:

$$TC^{TS}_{Tan}(C_n, t_a, t_b) = \sum_{p\ vertex\ of\ C_n([t_a,t_b])} \beta^{TS}_{dev}(p).$$

Propositions 3 and 4 imply that:

$$\begin{aligned} TC^{TS}_{Tan}(C_n, t_a, t_b) &\leq K_1\ (2\alpha_n)^2\ \sharp(C_n, t_a, t_b) && \text{(by Prop. 3)} \\ &\leq K_1\ K_2\ (2\alpha_n)^2\ \left[1 +\ TC^{TS}_{Tan}(C_n, t_a, t_b) + 2^n\ l(C_n, t_a, t_b)\right] && \text{(by Prop. 4)} \\ &\leq \frac{4K^2}{4^n}\ K_1\ K_2\ \left[1 + TC^{TS}_{Tan}(C_n, t_a, t_b)\ + 2^n\ l(C_n, t_a, t_b)\ \right]. \end{aligned}$$

Therefore, for some constants K_3 and K_4 independant of t_a and t_b, one has:

$$TC_{Tan}^{TS}(C_n, t_a, t_b) \leq \frac{K_3}{2^n} l(C_n, t_a, t_b) + \frac{K_4}{4^n}. \tag{28}$$

Step 2: We have:

$$\begin{aligned} TC_{3D}(C_n, t_a, t_b) &= \sum_{p \text{ vertex of } C_n(t_a,t_b)} \beta_{dev}^{3D}(p) \\ &\leq K_1 \ (2\alpha_n) \ \sharp (C_n, t_a, t_b) && \text{(By Prop. 3)} \\ &\leq 2K_1 \ K_2 \ \alpha_n \ \left[1 + \ TC_{Tan}^{TS}(C_n, t_a, t_b) + \ 2^n l(C_n, t_a, t_b)\right] && \text{(By Prop. 4)} \\ &\leq \tfrac{2KK_1K_2}{2^n} \left[1 + \ TC_{Tan}^{TS}(C_n, t_a, t_b) + \ 2^n l(C_n, t_a, t_b)\right]. \end{aligned}$$

Equation (28) then implies that there exist constants K and K':

$$TC_{3D}(C_n, t_a, t_b) \leq K \ l(C_n, t_a, t_b) + K' \ \frac{1}{2^n}, \tag{29}$$

and Proposition 5 then implies that the curve C is of class $\mathcal{C}^{1,1}$.

Step 3

We consider a point $p_0 = C(t_0)$ and we recall that $P_{p_0}^S$ denotes the projection on the plane $\Pi_{p_0}^S$ tangent to S at p_0. In this step, we are are going to prove the following lemma:

Lemma 9 *For any $t_0 \in (0, 1)$, the projection $P_{C(t_0)}^S \circ C$ of C on the plane tangent to S at $C(t_0)$ is twice derivable at t_0 and:*

$$\left. \frac{d^2 \left(P_{C(t_0)}^S \circ C\right)}{dt^2} \right|_{t = t_0} = 0$$

Proof of Lemma 9

Let α be such that $0 < \alpha < 1$ and let $r > 0$ be the reach of the surface S. We recall the following proposition (see [11] page 435):

Proposition 6 *In the ball $B(p_0, \alpha r)$, the map ξ is $\frac{1}{1-\alpha}$−Lipschitz.*

Let s be such that $s\widetilde{l} = \frac{\alpha r}{2}$. Using the fact that C_n has uniform parametrization and length upper bounded by $\widetilde{l}$, the respective lengths of the arcs $C_n([t_0 - s, t_0])$ and $C_n([t_0, t_0 + s])$ are smaller than $\frac{\alpha r}{2}$ and therefore one has:

$$C_n([t_0 - s, t_0 + s]) \subset B\left(C_n(t_0), \frac{\alpha r}{2}\right).$$

Let us now take n large enough such that $\|C_n - C\|_\infty \leq \frac{\alpha r}{2}$. We then have:

$$C_n([t_0 - s, t_0 + s]) \subset B(p_0, \alpha r).$$

Using Proposition 6, it follows that the curves $\xi \circ C_n([t_0 - s, t_0])$ and $\xi \circ C_n([t_0, t_0 + s])$ have length bounded by $\frac{\alpha r}{1-\alpha}$. Therefore, since the curvature of S is bounded by $\frac{1}{r}$, if one denotes by $\Pi_{\xi(C_n(t))}^S$ the tangent planes to S at the point $\xi(C_n(t))$, one has, for any $t \in [t_0 - s, t_0 + s]$:

$$2 \ \sin \frac{\angle \left(\Pi_{\xi(C_n(t))}^S, \Pi_{\xi(C_n(t_0))}^S\right)}{2} \leq \frac{\alpha}{1-\alpha}. \tag{30}$$

We consider now the sequence of curves $\widetilde{C}_n$ which are the projections $P^S_{p_0}\left(C_n\right)$ and which converge toward the projection $P^S_{p_0} \circ C$ of the curve C on the plane $\Pi^S_{p_0}$.
We consider the arc curve $P^S_{p_0}\left(C\left([t_0 - s, t_0 + s]\right)\right)$ in the plane $\Pi^S_{p_0}$, for $s = \frac{\alpha r}{\tilde{l}}$. Let t_i^n be the parameter of the i^{th} vertex $p_i^n = C_n(t_i^n)$ of C_n.
We apply Lemma 3, taking the unit vectors along $\frac{dC_n}{dt}$ just before and just after the vertex p_i^n for the vector u and v of the proposition, and the projections $P^S_{\xi(p_i^n)}$ and $P^S_{p_0}$ respectively for the projections P_1 and P_2 of the proposition.
If $\beta^{3D}_{dev}(p_i^n)$, $\beta^{TS}_{dev}(p_i^n)$ and $\beta_i^{\Pi_0}$ are respectively the 3D deviation angle of C_n at the vertex $C_n(t_i^n)$, the 2D deviation angle of $P^S_{\xi(p_i^n)}\left(C_n\left([t_0 - s, t_0 + s]\right)\right)$ at $\xi(p_i^n)$ and the 2D deviation angle of $P^S_{p_0}\left(C_n\left([t_0 - s, t_0 + s]\right)\right)$ at $P_{p_0}\left(p_i^n\right)$, Lemma 3 gives, that for some constant K:

$$\beta_i^{\Pi_0} \leq K \left[\beta^{TS}_{dev}(p_i^n) + \sin\theta \, \beta^{3D}_{dev}(p_i^n) + \frac{1}{4^n}\right],$$

where θ is the angle between the planes $\Pi^S_{\xi(p_i^n)}$ and $\Pi^S_{p_0}$ and satisfies $\sin\theta \leq 2\sin\frac{\theta}{2} \leq \frac{\alpha}{1-\alpha}$ from Equation 30. One has then:

$$\beta_i^{\Pi_0} \leq K \left[\beta^{TS}_{dev}(p_i^n) + \frac{\alpha}{1-\alpha} \, \beta^{3D}_{dev}(p_i^n) + \frac{1}{4^n}\right].$$

Therefore, by summing over all the vertices p_i^n, one has:

$$\begin{aligned}
& TC_{3D}\left(P^S_{p_0} \circ C_n, t_0 - s, t_0 + s\right) \\
\leq \quad & K' \left[TC^{TS}_{Tan}(C_n, t_0 - s, t_0 + s) + \frac{\sharp\left(C_n, t_0 - s, t_0 + s\right)}{4^n}\right. \\
& \left. \qquad + \frac{\alpha}{1-\alpha} TC_{3D}\left(C_n, t_0 - s, t_0 + s\right)\right].
\end{aligned}$$

Equation (28) gives:

$$TC^{TS}_{Tan}(C_n, t_0 - s, t_0 + s) \leq \frac{K_3 \; 2s}{2^n} + \frac{K_4}{4^n}.$$

Equation (29) implies:

$$TC_{3D}\left(C_n, t_0 - s, t_0 + s\right) \leq K \; 2s\tilde{l} + K' \; \frac{1}{2^n}.$$

Proposition 4 implies that:

$$\sharp\left(C_n, t_0 - s, t_0 + s\right) \leq K_2 \left[1 + \; TC^{TS}_{Tan}(C_n, t_0 - s, t_0 + s) + 2^n \; 2s\tilde{l}\right].$$

Then, by combining all these results and using the fact that $\alpha = \frac{\tilde{l}s}{r}$, we have, for some constant K_5 , K_6 and K_7:

$$TC_{3D}\left(P^S_{p_0} \circ C_n, t_0 - s, t_0 + s\right) \leq \frac{K_5}{4^n} + K_6 \; \frac{s}{2^n} + K_7 \; s^2.$$

Let $t \in [t_0 - s, t_0 + s]$. Lemma 2 implies that there exists K_8 such that:

$$\left\| \frac{d\left(P_{p_0}^S \circ C_n\right)}{dt}(t) - \frac{d\left(P_{p_0}^S \circ C_n\right)}{dt}(t_0) \right\| \leq TC_{3D}\left(P_{p_0}^S \circ C_n, t_0 - s, t_0 + s\right) + \frac{K_8}{2^n}.$$

We finally have:

$$\left\| \frac{d\left(P_{p_0}^S \circ C_n\right)}{dt}(t) - \frac{d\left(P_{p_0}^S \circ C_n\right)}{dt}(t_0) \right\| \leq \frac{K_5}{4^n} + K_6 \ \frac{s}{2^n} + K_7 \ s^2 + \frac{K_8}{2^n}. \tag{31}$$

Let $\epsilon > 0$. By Lemma 8, there exists N such that for every $n > N$ and for every $u \in (0,1)$:

$$\left\| \frac{dC}{dt}(u) - \frac{dC_n}{dt^+}(u) \right\| < \epsilon.$$

We then have for every $u \in (0,1)$:

$$\left\| \frac{d\left(P_{p_0}^S \circ C\right)}{dt}(u) - \frac{d\left(P_{p_0}^S \circ C_n\right)}{dt^+}(u) \right\| \leq \left\| \frac{dC}{dt}(u) - \frac{dC_n}{dt^+}(u) \right\| < \epsilon.$$

By using Equation (31), we have:

$$\begin{aligned} \left\| \tfrac{d\left(P_{p_0}^S \circ C\right)}{dt}(t) - \tfrac{d\left(P_{p_0}^S \circ C\right)}{dt}(t_0) \right\| \quad & \leq 2\epsilon + \left\| \tfrac{d\left(P_{p_0}^S \circ C_n\right)}{dt}(t) - \tfrac{d\left(P_{p_0}^S \circ C_n\right)}{dt}(t_0) \right\| \\ & \leq 2\epsilon + \tfrac{K_5}{4^n} + K_6 \ \tfrac{s}{2^n} + K_7 \ s^2 + \tfrac{K_8}{2^n}. \end{aligned}$$

We finally get:

$$\forall t \in [t_0 - s, t_0 + s], \ \left\| \frac{d\left(P_{p_0}^S \circ C\right)}{dt}(t) - \frac{d\left(P_{p_0}^S \circ C\right)}{dt}(t_0) \right\| \leq K_7 s^2.$$

This allows to conclude the proof. □

<u>Step 4</u>

Let $t_0 \in (0,1)$. In a neighborhood of $C(t_0)$, the surface S can be parametrized by its tangent plane at $C(t_0)$: in an appropriated frame with origin $C(t_0)$, the surface is parametrized by $(x,y) \in U_0 \mapsto (x, y, f(x,y))$, where U_0 is a neighborhood of $(0,0)$, f is a function of class $\mathcal{C}^2$ that satisfies $f(0,0) = 0$ and $Df(0,0) = 0$. For every t close enough to t_0, we put $\gamma(t) = P_{C(t_0)}^S(C(t))$. The function γ is of class $\mathcal{C}^{1,1}$, and by Lemma 9, we know that it is twice differentiable in t_0 and that $\gamma''(t_0) = 0$. We have in a neighborhood of t_0,

$$C(t) = \begin{pmatrix} \gamma(t) \\ f(\gamma(t)) \end{pmatrix}.$$

The function C is then twice differentiable in t_0 and we have

$$C''(t_0) = \begin{pmatrix} \gamma''(t_0) \\ D^2 f(\gamma(t_0)).(\gamma'(t_0), \gamma'(t_0)) + Df(\gamma(t_0).\gamma''(t_0) \end{pmatrix}.$$

The vector $C'(t_0)$ belongs to the tangent plane of S at $C(t_0)$. Furthermore, the fact that $Df(0,0) = 0$ implies that

$$D^2 f(\gamma(t_0)).(\gamma'(t_0), \gamma'(t_0)) = II_{C(t_0)}(C'(t_0)),$$

where $II_{C(t_0)}$ is the second fundamental form of S at the point $C(t_0)$ [7]. We then have

$$C''(t_0) = II_{C(t_0)}(C'(t_0))\ N^S_{C(t_0)}.$$

In this expression, $C''(t_0)$ depends continuously on t_0. That implies that C is of class $\mathcal{C}^2$ in t_0. The function C is then of class $\mathcal{C}^2$. Lemma 9 then implies that C has zero geodesic curvature, and then is a geodesic [7].

6.7 Proof of Corollary 1

We first need to check that the assumptions *a)* to *e)* of Theorem 1 are satisfied. The uniform convergence of P_n to f clearly implies assumption *a)*. Now, since the map ξ realises the distance to S, for every $m = P_n(u,v) \in P_n$, one has $\|\xi(m) - m\| \le \|f(u,v) - P_n(u,v)\|$ which implies assumption *b)*.
By using the regularity of f and by compacity, we have that:

$$m = \min_{(u,v)\in[0,1]^2} \left(\frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial v}(u,v), \frac{\partial f}{\partial u}(u,v) + \frac{\partial f}{\partial v}(u,v) \right) > 0,$$

and

$$m = \max_{(u,v)\in[0,1]^2} \left(\frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial v}(u,v), \frac{\partial f}{\partial u}(u,v) + \frac{\partial f}{\partial v}(u,v) \right) < \infty.$$

Let Δ_n be a triangle of P_n. The vertices of Δ_n are for example of the form

$$p_n = P_n\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \quad q_n = P_n\left(\frac{i+1}{2^n}, \frac{j}{2^n}\right) \quad \text{and} \quad r_n = P_n\left(\frac{i+1}{2^n}, \frac{j+1}{2^n}\right).$$

Now the fact that $(P_n)_{n\in\mathbb{N}}$ uniformly converges in derivative to f implies that there exists N and K such that for every $n > N$, the lengths of p_nq_n, p_nr_n and q_nr_n are greater than $\frac{1}{2^n}\left(m - \frac{K}{2^n}\right)$ and less than $\frac{1}{2^n}\left(M + \frac{K}{2^n}\right)$. If N is large enough we then have:

$$\begin{array}{c} \frac{1}{2^n}\frac{m}{2} \le p_nq_n \le \frac{1}{2^n}2\ M, \\ \frac{1}{2^n}\frac{m}{2} \le p_nr_n \le \frac{1}{2^n}2\ M, \\ \frac{1}{2^n}\frac{m}{2} \le r_nq_n \le \frac{1}{2^n}2\ M, \end{array} \tag{32}$$

In particular, assumption *d)* is proved.
The fact that P_n uniformly converges in derivative to f also implies that there exists N and K such that for every $n > N$:

$$\left\| 4^n \overrightarrow{q_nr_n} \wedge \overrightarrow{q_np_n} - \frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v}\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \right\| \le \frac{K}{2^n}. \tag{33}$$

If we take N such that $\frac{K}{2^n} \leq \frac{m}{2}$, we then have

$$\|4^n \overrightarrow{q_n r_n} \wedge \overrightarrow{q_n p_n}\| \geq m - \frac{K}{2^n} \geq \frac{m}{2}.$$

Together with Equation (32), that implies that

$$\sin \angle(\overrightarrow{q_n p_n}, \overrightarrow{q_n r_n}) = \frac{\|\overrightarrow{q_n p_n} \wedge \overrightarrow{q_n r_n}\|}{q_n p_n \ q_n r_n} \geq \frac{\frac{m}{2}\frac{1}{4^n}}{\left(\frac{1}{2^n} 2\ M\right)^2} = \frac{m}{8M} > 0.$$

The angle $\angle(\overrightarrow{q_n p_n}, \overrightarrow{q_n r_n})$ is then lower bounded by a constant independant on n. The same result holds with the other angles of T_n, which proves assumption *e)*.
We now denote by $N_{q_n}^{T_n}$ a unitary normal of the triangle $p_n q_n r_n$ and $N_{i,j}^S$ the normal of S at $f(\frac{i}{2^n}, \frac{j}{2^n})$. By using Equation (33) and Lemma 2, we have:

$$\begin{aligned} 2\ \sin \frac{\angle\left(N_{q_n}^{T_n}, N_{i,j}^S\right)}{2} &\leq \frac{\left\|4^n \overrightarrow{q_n r_n} \wedge \overrightarrow{q_n p_n} - \frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial u}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right\|}{\min\left(4^n \|\overrightarrow{q_n r_n} \wedge \overrightarrow{q_n p_n}\|,\ \left\|\frac{\partial f}{\partial u}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right\|\right)} \\ &\leq \frac{\frac{K}{2^n}}{\min\left(\left(\frac{m}{2}\right)^2 \sin \theta_{min}, m\right)}. \end{aligned}$$

Let now $m_n \in \Delta_n$. Then we need to bound the angle $\angle(N_{q_n}^{T_n}, N_{\xi(m_n)}^S)$.
There exists $\tilde{K}$ such that

$$\begin{aligned} & \left\|\xi(m_n) - f\left(\tfrac{i}{2^n}, \tfrac{j}{2^n}\right)\right\| \\ \leq\ & \|\xi(m_n) - m_n\| + \left\|m_n - P_n\left(\tfrac{i}{2^n}, \tfrac{j}{2^n}\right)\right\| + \left\|P_n\left(\tfrac{i}{2^n}, \tfrac{j}{2^n}\right) - f\left(\tfrac{i}{2^n}, \tfrac{j}{2^n}\right)\right\| \\ \leq\ & \tfrac{\tilde{K}}{2^n}. \end{aligned}$$

Then, by using Proposition 6, we have:

$$\|N_{\xi(m_n)}^S - N_{i,j}^S\| \leq \rho \ \frac{1}{1 - \frac{\tilde{K}}{2^n}} \ \frac{\tilde{K}}{2^n},$$

which implies that there exists k such that:

$$\angle(N_{q_n}^{T_n}, N_{\xi(m_n)}^S) \leq \angle\left(N_{q_n}^{T_n}, N_{i,j}^S\right) + \angle(N_{\xi(m_n)}^S, N_{i,j}^S) \leq \frac{k}{2^n}.$$

This result holds for all the triangles of T_n. Assumption *c)* is then proved.
We now only need to prove that $\xi(C_n)$ is interior to S for n large enough: the curve C is interior to S. The compacity of S and C implies that the distance from C to the boundary of S is more than $\eta > 0$. The curve $\xi(C_n)$ clearly tends to C. That implies that for n large enough $\xi(C_n)$ is an interior curve.

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8 Conclusion and future works

The main result of this work gives sufficient conditions for a sequence of geodesics on PL-surfaces to converge toward a geodesic on a smooth limit surface. We believe this is a significant step toward an effective notion of geodesic: indeed, the usual definition of geodesic is not effective because it relies on the notions of smooth curves and surfaces and on the pointwise curvature which can not be exactly represented on computers. Our main theorem states that the usual notion of geodesic coincides with the limit of a sequence of PL-curves that can be represented (at least if one restricts ourselves to PL-surfaces with rational vertices coordinates). Therefore, by using our result, a realistic algorithm can output a sequence of curves whose limit is a geodesic of a smooth surface. Notice that, given a smooth surface with bounded curvature, there exists a sequence of PL-surfaces converging to it (and that matches the conditions of our theorem). However, in order to completely get the effective notion of geodesic, one still has to quantify the rate of convergence of this sequence of curves.

We also believe that our result could be improved by relaxing the condition on the edge lengths: indeed, in the counter-example the lengths decrease with the order $\frac{1}{4^n}$ with respect to a decrease rate of $\frac{1}{2^n}$ of the angular convergence. We believe that it is possible to improve the theorem between the $\frac{K}{2^n}$ condition of the theorem and the $\frac{1}{4^n}$ of the counter-example.

Another possible improvement of the result is to suppose that the limit surface is of class $\mathcal{C}^{1,1}$ (instead of $\mathcal{C}^2$). Notice that such a generalisation would be very usefull for some subdivision surfaces with extraordinary points. Indeed, at extraordinary points, the limit surface of some subdivision surfaces is only of class $\mathcal{C}^{1,1}$. We proved (for example for the Catmull-Clark scheme) that if the limit curve of a sequence of geodesics does not contain extraordinary points, then it is a geodesic. We believe that the result still holds if the limit curve contains extraordinary points.

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