Finding Sums of Powers*

1 Recursive procedure

Consider:

$$\begin{aligned}(N+1)\begin{bmatrix}1^k\\2^k\\\vdots\\N^k\end{bmatrix} &= \begin{bmatrix}1^k & & & & & \\ 2^k & + & 2^k & & & \\ \vdots & & & \vdots & & \\ N^k & + & N^k & \dots & + & N^k\end{bmatrix} + \begin{bmatrix}1^k & \dots & + & 1^k\\ & \dots & + & 2^k\\ & \vdots & & \vdots\\ & & & N^k\end{bmatrix}\\ &= \begin{bmatrix}1^{k+1}\\2^{k+1}\\\vdots\\N^{k+1}\end{bmatrix} + \left(\begin{bmatrix}1^k\\ \\ \\ \end{bmatrix} + \begin{bmatrix}1^k\\2^k\\ \\ \end{bmatrix} + \dots + \begin{bmatrix}1^k\\2^k\\\vdots\\N^k\end{bmatrix}\right)\end{aligned} \tag{1}$$

This shows that:

$$(N+1)\sum_{i=1}^{N} i^k = \sum_{i=1}^{N} i^{k+1} + \sum_{j=1}^{N}\sum_{i=1}^{j} i^k \tag{2}$$

Thus:

- From the formulae in N for $\sum_{i=1}^{N} i^j$, for $j \leq k$, one derive a formula for $\sum_{i=1}^{N} i^{k+1}$, for instance:
  1. From $\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$ twice,
  2. we have: $(N+1)\left(\frac{1}{2}N(N+1)\right) = \sum_{i=1}^{N} i^2 + \sum_{j=1}^{N}\left(\frac{1}{2}j(j+1)\right) = \sum_{i=1}^{N} i^2 + \frac{1}{2}\sum_{j=1}^{N} j^2 + \frac{1}{2}\sum_{j=1}^{N} j$;
  3. thus with the previous sum again: $\frac{3}{2}\sum_{i=1}^{N} i^2 = \frac{1}{2}N(N+1)^2 - \frac{1}{2}\left(\frac{1}{2}N(N+1)\right)$;
  4. which is of course: $\sum_{i=1}^{N} i^2 = \frac{1}{3}N(N+1)(N-\frac{1}{2})$.
- Proofs by induction, such as: $\sum_{i=1}^{N} i^k = \frac{1}{k+1}N^{k+1} + \mathcal{O}(N^k)$.

2 Pascal triangle

Develop $(j+1)^{k+1}$, for any j, as $(j+1)^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}j^i = j^{k+1} + \sum_{i=0}^{k}\binom{k+1}{i}j^i$.

Then we have also

$$(j+1)^{k+1} - j^{k+1} = \sum_{i=0}^{k}\binom{k+1}{i}j^i. \tag{3}$$

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Summing for $j = 1..N$ the telescoping terms, we get that the first sums are combined by the Pascal triangle:

$$\sum_{j=1}^{N}(j+1)^{k+1} - j^{k+1} = (N+1)^{k+1} - 1 = \sum_{i=0}^{k}\binom{k+1}{i}\left(\sum_{j=1}^{N} j^i\right). \tag{4}$$

Let $S_k(N) = \sum_{i=0}^{N} i^k$. Equation (4) also gives: $(N+1)^{k+1} = (S_0(N)+1) + \sum_{i=1}^{k}\binom{k+1}{i}S_i(N)$.
For instance, this is:

$$\begin{bmatrix} N+1 \\ (N+1)^2 \\ (N+1)^3 \\ (N+1)^4 \\ (N+1)^5 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 2 & & & \\ 1 & 3 & 3 & & \\ 1 & 4 & 6 & 4 & \\ 1 & 5 & 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} 1+S_0(N) \\ S_1(N) \\ S_2(N) \\ S_3(N) \\ S_4(N) \end{bmatrix} \tag{5}$$

From which we can obtain after transposition:

$$S_4(N) = \begin{bmatrix} (N+1) & (N+1)^2 & (N+1)^3 & (N+1)^4 & (N+1)^5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 2 & 3 & 4 & 5 \\ & & 3 & 6 & 10 \\ & & & 4 & 10 \\ & & & & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

Backsubstituing, we get:

$$\begin{aligned} S_4(N) &= \begin{bmatrix} (N+1) & (N+1)^2 & (N+1)^3 & (N+1)^4 & (N+1)^5 \end{bmatrix} \begin{bmatrix} -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \end{bmatrix}^T \\ &= -\frac{1}{30}(N+1) + \frac{1}{3}(N+1)^3 - \frac{1}{2}(N+1)^4 + \frac{1}{5}(N+1)^5 \\ &= -\frac{1}{30}N + \frac{1}{3}N^3 + \frac{1}{2}N^4 + \frac{1}{5}N^5 \end{aligned} \tag{7}$$

More generally, this gives the following backsubstitution algorithm for the computation of a polynomial formula for the sum of powers:

$$S_k(N) = \begin{bmatrix} (N+1) & (N+1)^2 & \dots & (N+1)^{k+1} \end{bmatrix} \begin{bmatrix} \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \dots & \binom{k+1}{1} \\ & \binom{2}{2} & \binom{3}{2} & \dots & \binom{k+1}{2} \\ & & \ddots & & \vdots \\ & & & \ddots & \binom{k+1}{k} \\ & & & & \binom{k+1}{k+1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

Using the Bernoulli numbers B_j (with the convention that $B_1 = +\frac{1}{2}$), one could also get an explicit formula, the Faulhaber's formula:

$$S_k(N) = \sum_{i=1}^{N} i^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j N^{k-j+1}. \tag{9}$$

For more on Faulhaber's formula, Pascal's triangle and sums of powers, see, e.g., [1].

References

[1] Anthony William F Edwards. Sums of powers of integers: a little of the history. *The Mathematical Gazette*, 66(435):22–28, 1982.