Analytic parametrization and volume minimization of three dimensional bodies of constant width

Térence Bayen, Thomas Lachand-Robert & Édouard Oudet*

v0.24 – April 6, 2006

Abstract

We present a complete analytic parametrization of constant width bodies in dimension 3 based on the median surface: more precisely, we define a bijection between some space of functions and constant width bodies. We compute simple geometrical quantities like the volume and the surface area in terms of those functions. As a corollary we give a new algebraic proof of Blaschke's formula. Finally, we derive weak optimality conditions for convex bodies which minimize the volume among constant width bodies.

1 Introduction

A body (that is, a compact connected subset K of $\mathbb{R}^n$) is said to be of *constant width* α if its projection on any straight line is a segment of length $\alpha \in \mathbb{R}_+$, the same value for all lines. This can also be expressed by saying that the *width map*

$$w_K : \nu \in \mathbf{S}^{n-1} \longmapsto \max_{x\in K} \nu \cdot x - \min_{x\in K} \nu \cdot x \tag{1}$$

has constant value α. This is also equivalent to the geometrical fact that two parallel support hyperplanes on K are always separated by a distance α, independent of their direction.

*Laboratoire de mathématiques, Université de Savoie, Campus scientifique, 73376 Le Bourget-du-lac, France. bayen@clipper.ens.fr, Thomas.Lachand-Robert@univ-savoie.fr, Edouard.Oudet@univ-savoie.fr
http://www.lama.univ-savoie.fr/~lachand, http://www.lama.univ-savoie.fr/~oudet, http://www.lama.univ-savoie.fr/~bayen.

Obvious bodies of constant width are the balls; but they are many others. These bodies, also called *orbiforms* in dimension two, or *spheroforms* in dimension three (as in [2]), have many interesting properties and applications. Orbiforms in particular have been studied a lot during the nineteenth century and later, particularly by Frank Reuleaux, whose name is now attached to those orbiforms you get by intesecting a finite number of disks of equal radii α, whose center are vertices of a regular polygon of diameter α.

Among the oldest problems related to these bodies of constant width are the question of which are those with maximal or minimal volume, for a given value of the width α. It is not difficult to prove that the ball (of radius $\alpha/2$) has maximal volume: this follows from the isoperimetric inequality.

On the other hand, the question of which body of constant width α has minimal volume proved to be much more difficult. First notice that this problem is not correctly stated: indeed, one can remove the interior of a body to decrease its volume, without changing its constant width property. Therefore, we need to add an additional requirement for the problem to make sense (even though this is not needed for the maximization problem). The problem is well-posed if we consider only *convex* bodies, and this is the usual statement considered.

So let us define formally the following class:

$$\mathcal{W}_\alpha := \big\{K \subset \mathbb{R}^n \,;\, K \text{ compact convex and } \forall \nu \in \mathbf{S}^{n-1},\ w_K(\nu) = \alpha\big\}. \tag{2}$$

The problem of interest is now to minimize the n-dimensional volume, denoted by $|K|$ hereafter:

$$\text{Find } K^* \in \mathcal{W}_\alpha \text{ such that } |K^*| = \min_{K \in \mathcal{W}_\alpha} |K| \,. \tag{3}$$

Note that the existence of K^* is easy to establish. Indeed $\mathcal{W}_\alpha$ is a compact class of sets for most reasonable topologies (for instance the Haussdorff topology), and the volume is a continuous function.

In dimension two, the problem was solved by Lebesgue and Blaschke: the solution turns out to be a *Reuleaux triangle*.

In dimension three, the problem is still open. Indeed the mere existence of non trivial three-dimensional bodies of constant width is not so easy to establish. In particular, no finite intersection of balls has constant width (except balls themselves), a striking difference with the two-dimensional case.

A simple construction is to consider a two dimensional body of constant width having an axis of symmetry (like the Reuleaux triangle for instance): the corresponding body of revolution obtained by rotation around this axis is a spheroform. F. Meissner proved that the *rotated Reuleaux triangle* has the smaller volume among bodies of revolution in $\mathcal{W}_\alpha$.

Later on he was able to construct another spheroform (usually called "Meissner's tetrahedron") which does not have the symmetry of revolution. The volume of this body is smaller than any other known of constant width, so it is a good candidate as a solution to the problem (3). We describe this body in more details later on in this paper. Let us just say for the moment that it looks like an intersection of four balls centered on the vertices of a regular tetrahedron, but some of the edges are smoothed; in particular, it doesn't have all the symmetries of a regular tetrahedron.

In this paper we first present a complete analytic parametrization of constant width bodies in dimension 3 based on the median surface. More precisely, we define a bijection between the space of functions $\mathrm{C}^{1,1}_\sigma(\Omega)$ and constant width bodies. Then, we compute simple geometrical quantities like the volume and the surface area in terms of those functions. As a corollary we give a new algebraic proof of Blaschke's formula and compute the surface and the volume of Meissner's tetrahedron. Finally, we derive weak optimality conditions for the problem (3).

2 The Median Surface

In this section, we introduce a geometrical tool, which we call the *median surface*.

2.1 Definition and basics

For a convex body K, we say that a hyperplane H is a *hyperplane of support for K at x*, if $x \in K \cap H$ and K is included in one of the half-spaces limited by H. If $\nu \in \mathbf{S}^{n-1}$ is a normal vector to H, pointing outside the half space containing K, we say that ν is *an outward support vector at x*. Obviously if K is smooth (that is, has a differentiable boundary), then ν is just the outward unit normal at x. In this particular case, there is a map $x \mapsto \nu$ which is usually called *the Gauss map*.

The reverse Gauss map (which is well defined for a body of constant see for instance [11]), satisfies $R_K(\nu) - R_K(-\nu) = \alpha\nu$ for all ν. We may now introduce a parallel surface to ∂K. Consider, for all $\nu \in \mathbf{S}^{n-1}$ the point,

$$M_K(\nu) := R_K(\nu) - \frac{\alpha}{2}\nu = R_K(-\nu) + \frac{\alpha}{2}\nu.$$

Notice that $M_K(-\nu) = M_K(\nu)$. The set of points $M_K(\nu)$ is called the *median surface* of the body K.

Let us recall from [11] one geometrical characterization of constant width bodies:

Theorem 1 *Let K be closed subset of $\mathbb{R}^n$. Then K has constant width α if and only if it satisfies:*

$$\forall \nu \in \mathbf{S}^{n-1},\ \exists x_\nu \in K,$$
$$x_\nu + \alpha\nu \in K \quad \text{and} \quad \forall y \in K,\ (y - x_\nu)\cdot\nu \in [0,\alpha]. \tag{4}$$

2.2 Construction of constant width sets

We present in this section a construction process of constant width bodies starting from an appropriate surface, which will be their median surface. More precisely:

Theorem 2 *Let $\alpha > 0$ be given and $M : \mathbf{S}^{n-1} \to \mathbb{R}^n$ be a continuous application satisfying*

$$\forall \nu \in \mathbf{S}^{n-1}, \qquad M(-\nu) = M(\nu); \tag{5}$$

$$\forall \nu_0, \nu_1 \in \mathbf{S}^{n-1}, \qquad \big(M(\nu_1) - M(\nu_0)\big)\cdot\nu_0 \le \frac{\alpha}{4}\,|\nu_1 - \nu_0|^2. \tag{6}$$

Define a subset $K \subset \mathbb{R}^n$ as follows:

$$K := \Big\{M(\nu) + t\nu\,;\, \nu \in \mathbf{S}^{n-1}, t \in \Big[0, \frac{\alpha}{2}\Big]\Big\}. \tag{7}$$

Then K is a convex body of constant width α, and $M_K \equiv M$.

Conversely, any convex body of constant width α can be described by (7)*, where $M = M_K$.*

Notice that we could have defined K by

$$K := \Big\{M(\nu) + t\nu\,;\, \nu \in \mathbf{S}^{n-1}, t \in \Big[-\frac{\alpha}{2}, \frac{\alpha}{2}\Big]\Big\}. \tag{8}$$

This is equivalent to (7), due to (5). Similarly, taking (5) into consideration, we can rewrite (6) with $-\nu_0, -\nu_1$. We deduce that for an application M satisfying (5), (6) is equivalent to:

$$\forall \nu_0, \nu_1 \in \mathbf{S}^{n-1}, \qquad \big|\big(M(\nu_1) - M(\nu_0)\big)\cdot\nu_0\big| \le \frac{\alpha}{4}\,|\nu_1 - \nu_0|^2. \tag{9}$$

In order to prove this theorem, we make use of a lemma:

Lemma 1 *Under the assumptions of Theorem 2, let K be defined by* (7)*. Then $\mathbb{R}^n = \{M(\nu) + t\nu\,;\, \nu \in \mathbf{S}^{n-1}, t \in \mathbb{R}_+\}$, K is compact, and*

$$\partial K \subset \Big\{M(\nu) + \frac{\alpha}{2}\nu\,;\, \nu \in \mathbf{S}^{n-1}\Big\}. \tag{10}$$

(It will come from Theorem 2 that there is actually equality for the sets in (10).)

Proof. Consider the map $Q : \mathbf{S}^{n-1} \times \mathbb{R} \mapsto M(\nu) + t\nu$ where M satisfies (5). Since M is continuous, $K = Q\left(\mathbf{S}^{n-1} \times [0, \frac{\alpha}{2}]\right)$ is a compact set.

Let us first prove that $Q(\mathbf{S}^{n-1} \times \mathbb{R}_+) = \mathbb{R}^n$. Note that $Q(\mathbf{S}^{n-1} \times \mathbb{R}_+) = Q(\mathbf{S}^{n-1} \times \mathbb{R})$ from (5). We consider some $x \in \mathbb{R}^n$, and assume by contradiction that $x \notin Q(\mathbf{S}^{n-1} \times \mathbb{R})$. For each ν, define x_ν as the projection of x onto the straight line $M(\nu) + \mathbb{R}\nu$. Our assumption implies $x \neq x_\nu$. Moreover

$$x_\nu = M(\nu) + t_\nu \nu \quad \text{where} \quad t_\nu := \nu \cdot \big(x - M(\nu)\big)$$

as a classical property of the projection.

Since in particular $x \neq M(\nu)$ for all ν, we can define a map $f : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ by $f(\nu) := (x - M(\nu))/\left|x - M(\nu)\right|$. Note that f is continuous, and $f(-\nu) = f(\nu)$. Such a map has an even topological degree, and in particular has a fixed point [5]. Therefore there exists some ν such that $f(\nu) = \nu$. For such a ν, we get $x_\nu = x$, a contradiction.

We now turn on the proof of (10). Consider some $x \in \partial K$. In particular, $x \in K$, so $x = M(\nu_0) + t_0\nu_0$ for some ν_0 and $t_0 \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]$. There exists a sequence $(x_n) \subset \mathbb{R}^n \setminus K$ with limit x. From our previous study, we known that $x_n = M(\nu_n) + t_n\nu_n$ for some $\nu_n \in \mathbf{S}^{n-1}$ and $t_n \in \mathbb{R}_+$. The assumption $x_n \notin K$ implies $t_n > \alpha/2$, but on the other hand the sequence (t_n) is bounded since (x_n) is bounded and $M(\mathbf{S}^{n-1})$ is compact. Therefore we may assume that the sequences (ν_n) and (t_n) are convergent. Let us denote by ν_∞ and $t_\infty \geq \frac{\alpha}{2}$ their limits. Since M is continuous, we have $x = M(\nu_\infty) + t_\infty\nu_\infty$.

In particular, $M(\nu_0) = x - t_0\nu_0 = M(\nu_\infty) + t_\infty\nu_\infty - t_0\nu_0$. Let us assume with no loss of generality that $\nu_0 \cdot \nu_\infty \geq 0$ (otherwise we just have to change ν_0 to $-\nu_0$ and t_0 to $-t_0$). We write (6) for ν_∞, ν_0, so

$$\begin{aligned}
& \big(M(\nu_0) - M(\nu_\infty)\big) \cdot \nu_\infty \leq \frac{\alpha}{4}\left|\nu_\infty - \nu_0\right|^2 = \frac{\alpha}{2}(1 - \nu_0 \cdot \nu_\infty) \\
\iff \quad & t_\infty - t_0\,\nu_0 \cdot \nu_\infty \leq \frac{\alpha}{2}(1 - \nu_0 \cdot \nu_\infty) \\
\iff \quad & t_\infty \leq \frac{\alpha}{2} - \left(\frac{\alpha}{2} - t_0\right)\nu_0 \cdot \nu_\infty \leq \frac{\alpha}{2}
\end{aligned}$$

since $t_0 \in \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]$. This proves that $t_\infty = \frac{\alpha}{2}$. Hence $x \in Q(\mathbf{S}^{n-1}, \frac{\alpha}{2})$. $\square$

Proof of Theorem 2. We begin with the proof of the reciprocal statement in the Theorem. Let K be a body of constant width. We already know that its median surface $M = M_K$ is continuous and satisfies (5). Since $M_K(\nu) = R_K(\nu) - \frac{\alpha}{2}\nu$, and $R_K(\nu_1) \cdot \nu_0 \leq R_K(\nu_0) \cdot \nu_0$ from the definition of R_K, we have

$$\big(M(\nu_1) - M(\nu_0)\big) \cdot \nu_0 \leq \frac{\alpha}{2}(1 - \nu_0 \cdot \nu_1) = \frac{\alpha}{4}\left|\nu_1 - \nu_0\right|^2.$$

This proves (6).

Since K is convex and $M_K(\nu) + \frac{\alpha}{2}\nu \in K$, $M_K(\nu) - \frac{\alpha}{2}\nu \in K$, we see that K contains the right hand side of (8). Now let $x \in K$ be given, and let y be the farthest point from x in K. Define $\nu := (y-x)/|y-x|$. For any $z \in K$, we have

$$y \cdot \nu = |y-x| + x \cdot \nu \geq |z-x| + x \cdot \nu \geq (z-x) \cdot \nu + x \cdot \nu = z \cdot \nu$$

so $y = R_K(\nu) = M_K(\nu) + \frac{\alpha}{2}\nu$. Hence $x = M_K(\nu) + t\nu$ with $t = \frac{\alpha}{2} - |x-y|$. Since $|x-y| \leq \alpha$, we have $|t| \leq \frac{\alpha}{2}$, which concludes the proof of (8).

We now prove the direct statement in the Theorem. So consider a map M satisfying (5) and (6), and K be defined by (7) (or (8) equivalently). In view of Theorem 1, we need to prove (4).

Let $\nu \in \mathbf{S}^{n-1}$ be given. Consider $x_\nu := M(\nu) - \frac{\alpha}{2}\nu$, so that $x_\nu + \alpha\nu = M(\nu) + \frac{\alpha}{2}\nu \in K$ from its definition.

Consider any $y \in K$, so that $y = M(\hat{\nu}) + t\hat{\nu}$. Changing $\hat{\nu}$ and t to their opposite, if necessary, we may assume that $\nu \cdot \hat{\nu} \geq 0$. Note that

$$(y - x_\nu) \cdot \nu = \big(M(\hat{\nu}) - M(\nu)\big) \cdot \nu + t\nu \cdot \hat{\nu} + \frac{\alpha}{2}.$$

Using (9) with $\nu_0 = \nu, \nu_1 = \hat{\nu}$, we get

$$-\frac{\alpha}{2}(1 - \nu \cdot \hat{\nu}) \leq \big(M(\hat{\nu}) - M(\nu)\big) \cdot \nu \leq \frac{\alpha}{2}(1 - \nu \cdot \hat{\nu}).$$

Hence, since $t \in \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]$:

$$0 \leq (t + \frac{\alpha}{2})\ \nu \cdot \hat{\nu} \leq (y - x_\nu) \cdot \nu \leq \alpha + (t - \frac{\alpha}{2})\ \nu \cdot \hat{\nu} \leq \alpha.$$

This concludes the proof of the theorem. $\square$

Applications M satisfying (5) and (6) will play an important role in the remaining of this paper. So let us give a few additionnal properties on them. We start here with simple inequalities, and will consider what happens on a differential level in the next section. Note that all these results apply in particular to the median surface of any convex body of constant width according to Theorem 2.

Lemma 2 *Let M be a continuous application satisfying* (5) *and* (6)*. Then M is $\frac{\alpha}{2}$-lipschitzian:*

$$\forall \nu_0, \nu_1 \in \mathbf{S}^{n-1}, \qquad |M(\nu_1) - M(\nu_0)| \leq \frac{\alpha}{2}\,|\nu_1 - \nu_0|\,. \tag{11}$$

and satisfies:

$$\forall \nu_0, \nu_1 \in \mathbf{S}^{n-1}, \qquad \left| M(\nu_1) + \frac{\alpha}{2}\nu_1 - M(\nu_0) - \frac{\alpha}{2}\nu_0 \right| \leq \alpha. \tag{12}$$

Proof. According to Theorem 2, M is the median surface of some $K \in \mathcal{W}_\alpha$ defined by (7). Since K contains $M(\nu) + \frac{\alpha}{2}\nu$ for any ν, and has diameter α, we get (12).

Squaring the left hand side of (12) and expanding it, we get

$$|M(\nu_1) - M(\nu_0)|^2 - \alpha(M(\nu_1) - M(\nu_0)) \cdot (\nu_1 - \nu_0) \leq \frac{\alpha^2}{4} |\nu_1 + \nu_0|^2 \tag{13}$$

since $|\nu_1 - \nu_0|^2 + |\nu_1 + \nu_0|^2 = 4$. The above relation is true for any pair of unit vectors, so we can write it for $(\nu_1, -\nu_0)$ and $(-\nu_1, \nu_0)$. We get, taking (5) into account:

$$|M(\nu_1) - M(\nu_0)|^2 - \alpha(M(\nu_1) - M(\nu_0)) \cdot (\nu_1 + \nu_0) \leq \frac{\alpha^2}{4} |\nu_1 - \nu_0|^2$$
$$|M(\nu_1) - M(\nu_0)|^2 + \alpha(M(\nu_1) - M(\nu_0)) \cdot (\nu_1 + \nu_0) \leq \frac{\alpha^2}{4} |\nu_1 - \nu_0|^2 .$$

Summing these relations yields (11). $\square$

2.3 Smooth median surface

In this section we reduce (6) to local differential properties. This is easy whenever M is differentiable, but requires more involved statements in the general case. Note that M will always be defined on the sphere $\mathbf{S}^{n-1}$, and if differentiable, its derivative $DM(\nu)$ is defined on the tangent space to the sphere at ν, which is simply $\nu^\perp := \{w \in \mathbb{R}^n \,;\, w \cdot \nu = 0\}$. In the following proposition, we consider C^2 maps $\widetilde{\nu} : [0,1] \to \mathbf{S}^{n-1}$, and $\dot{\widetilde{\nu}}$ is the derivative of $\widetilde{\nu}$. Notice that only the end point $\widetilde{\nu}(0)$ and the corresponding derivatives do matter.

Proposition 1 *Let $M : \mathbf{S}^{n-1} \to \mathbb{R}^n$ be given. Then M satisfies* (9) *if and only if it satisfies*

$$\forall \widetilde{\nu} \in \mathrm{C}^2([0,1]; \mathbf{S}^{n-1}),$$
$$\limsup_{t \xrightarrow{>} 0} \frac{1}{t^2} \left| \Big(M(\widetilde{\nu}(t)) - M(\widetilde{\nu}(0)) \Big) \cdot \widetilde{\nu}(0) \right| \leq \frac{\alpha}{4} \left| \dot{\widetilde{\nu}}(0) \right|^2 . \tag{14}$$

If M is differentiable, then (14) *is equivalent to*

$$\forall \nu_0 \in \mathbf{S}^{n-1}, \forall w \in \nu_0^\perp,$$
$$\nu_0 \cdot DM(\nu_0) w = 0 \quad \textit{and} \quad |w \cdot DM(\nu_0) w| \leq \frac{\alpha}{2} |w|^2 . \tag{15}$$

We will shorten (15) in the following by writing it $\nu_0 \cdot DM(\nu_0) = 0$ (as vectors) and $\pm DM(\nu_0) \leq \frac{\alpha}{2}\,\mathrm{Id}$ (as matrices). This expresses the fact that ν_0 is the normal vector to the surface $\nu_0 \mapsto M(\nu_0)$ at $M(\nu_0)$, and that the absolute values of the curvature radii does not exceed $\frac{\alpha}{2}$. (See also the parametric equivalent in the next section.)

Proof. Assume first that M satisfies (9). Let $\nu \in \mathrm{C}^2([0,1];\mathbf{S}^{n-1})$, and define $\nu_0 := \widetilde{\nu}(0)$ for short. Note that $|\widetilde{\nu}(t)|^2 = 1$ for all t, so

$$\forall t, \qquad \widetilde{\nu}(t) \cdot \dot{\widetilde{\nu}}(t) = 0 \quad \text{and} \quad \widetilde{\nu}(t) \cdot \ddot{\widetilde{\nu}}(t) = -\left|\dot{\widetilde{\nu}}(t)\right|^2. \tag{16}$$

In particular a Taylor expansion near $t = 0$ yields

$$\widetilde{\nu}(t) \cdot \nu_0 = \left(\nu_0 + t\dot{\widetilde{\nu}}(0) + \frac{t^2}{2}\ddot{\widetilde{\nu}}(0) + o(t^2)\right) \cdot \nu_0 = 1 - \frac{t^2}{2}\left|\dot{\widetilde{\nu}}(0)\right|^2 + o(t^2).$$

Using (9) with $\widetilde{\nu}(t)$ and ν_0, we get:

$$\left|\Big(M(\widetilde{\nu}(t)) - M(\nu_0)\Big)\cdot\nu_0\right| \leq \frac{\alpha}{2}(1 - \widetilde{\nu}(t) \cdot \nu_0) \leq \frac{\alpha}{4}t^2\left|\dot{\widetilde{\nu}}(0)\right|^2 + o(t^2).$$

Dividing by t^2, we get (14).

If M is differentiable and $w \in \nu_0^\perp$, consider $\widetilde{\nu}(t) := p_{\mathbf{S}^{n-1}}(\nu_0 + tw)$ where $p_{\mathbf{S}^{n-1}} : x \mapsto x/|x|$ is the projection on the sphere. So $\widetilde{\nu}(0) = \nu_0$ and $\dot{\widetilde{\nu}}(0) = w$. Hence we have

$$M(\widetilde{\nu}(t)) \cdot \nu_0 = M(\nu_0) \cdot \nu_0 + t\nu_0 \cdot DM(\nu_0)w + o(t)$$

so (14) clearly implies $\nu_0 \cdot DM(\nu_0)w = 0$.

Assume for a moment that M is twice differentiable and satisfies (14). We already know that $\nu_0 \cdot DM(\nu_0)w = 0$ for any ν_0 and any $w \in \nu_0^\perp$. Therefore we have, for any $\nu \in \mathrm{C}^2([0,1];\mathbf{S}^{n-1})$:

$$\forall t, \qquad 0 = \widetilde{\nu}(t) \cdot DM(\widetilde{\nu}(t))\dot{\widetilde{\nu}}(t).$$

Differentiating this relation with respect to t, we get

$$0 = \dot{\widetilde{\nu}}(t) \cdot DM(\widetilde{\nu}(t))\dot{\widetilde{\nu}}(t) + \widetilde{\nu}(t) \cdot D^2M(\widetilde{\nu}(t))(\dot{\widetilde{\nu}}(t), \dot{\widetilde{\nu}}(t))$$

since $\widetilde{\nu}(t) \cdot DM(\widetilde{\nu}(t)) = 0$. Considering $t = 0$ and $w := \dot{\widetilde{\nu}}(0) \in \nu_0^\perp$ yields

$$\forall w \in \nu_0^\perp, \qquad w \cdot DM(\nu_0)w = -\nu_0 \cdot D^2M(\nu_0)(w, w). \tag{17}$$

Therefore a Taylor expansion yields

$$\begin{aligned} M(\widetilde{\nu}(t)) \cdot \nu_0 &= M(\nu_0) \cdot \nu_0 + \frac{1}{2} t^2 \ \nu_0 \cdot D^2 M(\nu_0)(w,w) + o(t^2) \\ &= M(\nu_0) \cdot \nu_0 - \frac{1}{2} t^2 \ w \cdot DM(\nu_0) w + o(t^2). \end{aligned} \tag{18}$$

It is now clear that (14) implies (15).

If M is not twice differentiable, we use an approximation argument as follows. For any $\beta > \alpha$ and any $\varepsilon > 0$, there exists an approximating map $M_\varepsilon \in \mathrm{C}^2(\mathbf{S}^{n-1}, \mathbb{R}^n)$, such that

$$\|M - M_\varepsilon\|_{W^{1,\infty}(\mathbf{S}^{n-1};\mathbb{R}^n)} \leq \varepsilon \tag{19}$$

and M_ε satisfies (14) with α replaced by β. Hence M_ε satisfies (15), also with α replaced by β. Letting ε go to zero and using (19), we deduce that M satisfies (15) with α replaced by β. Since this holds for any $\beta > \alpha$, it holds for α as well.

Conversely, if M is differentiable and satisfies (15), let us prove that it satisfies (14). Using exactly the same approximation, we see that we just have to prove that for M twice differentiable. In such a case, (15) implies (17). Hence the Taylor expansion (18) holds true. This yields (14).

Let us now prove the reverse statement of the proposition, that is, a map M satisfying (14) also satisfies (9). Again it is enough to prove it for a twice differentiable map, for (19) implies in particular uniform convergence of M_ε to M.

So let us consider two vectors ν_0, ν_1 in $\mathbf{S}^{n-1}$ and prove (9). We consider a geodesic path $\widetilde{\nu} \in \mathrm{C}^2([0,1];\mathbf{S}^{n-1})$ such that $\widetilde{\nu}(0) = \nu_0$ and $\widetilde{\nu}(1) = \nu_1$. Such a path satisfies $\widetilde{\nu}(t) \in (\mathbb{R}\nu_0 + \mathbb{R}\nu_1)$, and $\nu_0 \cdot \dot{\widetilde{\nu}}(t) \leq 0$ for all t.

The function $f : t \mapsto \nu_0 \cdot M(\widetilde{\nu}(t))$ has derivative $f'(t) = \nu_0 \cdot DM(\widetilde{\nu}(t))\dot{\widetilde{\nu}}(t)$. Since $\widetilde{\nu}(t) \in (\mathbb{R}\nu_0 + \mathbb{R}\nu_1)$, we have

$$\nu_0 = (\nu_0 \cdot \widetilde{\nu}(t)) \ \widetilde{\nu}(t) + \frac{(\nu_0 \cdot \dot{\widetilde{\nu}}(t))}{\left|\dot{\widetilde{\nu}}(t)\right|^2} \ \dot{\widetilde{\nu}}(t).$$

Taking (15) and $\nu_0 \cdot \dot{\widetilde{\nu}}(t) \leq 0$ into account, we get

$$|f'(t)| = \frac{\left|\nu_0 \cdot \dot{\widetilde{\nu}}(t)\right|}{\left|\dot{\widetilde{\nu}}(t)\right|^2} \ \left|\dot{\widetilde{\nu}}(t) \cdot DM(\widetilde{\nu}(t))\dot{\widetilde{\nu}}(t)\right| \leq -\frac{\alpha}{2}(\nu_0 \cdot \dot{\widetilde{\nu}}(t)).$$

Therefore

$$\begin{aligned}\left|\big(M(\nu_1) - M(\nu_0)\big)\cdot\nu_0\right| &= |f(1) - f(0)| \\ &\le -\frac{\alpha}{2}\int_0^1 (\nu_0\cdot\dot{\tilde{\nu}}(t))\ \mathrm{d}t = \frac{\alpha}{2}(1-\nu_0\cdot\nu_1).\end{aligned}$$

This completes the proof of the proposition. $\square$

Remark 2.A. Observe that (14) is equivalent to

$$\forall\tilde{\nu}\in \mathrm{C}^2([0,1];\mathbf{S}^{n-1}),$$
$$\limsup_{t \xrightarrow{>} 0} \frac{1}{t^2}\left|\Big(M(\tilde{\nu}(t)) - M(\tilde{\nu}(0))\Big)\cdot\tilde{\nu}(t)\right| \le \frac{\alpha}{4}\left|\dot{\tilde{\nu}}(0)\right|^2. \qquad (20)$$

Indeed we just have to prove that for a smooth M again. Then we may rewrite (18) with ν_0 and $\tilde{\nu}(t)$ reversed:

$$M(\nu_0)\cdot\tilde{\nu}(t) = M(\tilde{\nu}(t))\cdot\tilde{\nu}(t) - \frac{1}{2}t^2\,\dot{\tilde{\nu}}(t)\cdot DM(\tilde{\nu}(t))\dot{\tilde{\nu}}(t) + o(t^2).$$

Since $w = \dot{\tilde{\nu}}(0) = \dot{\tilde{\nu}}(t) + O(t)$ and DM is continuous, we get by substracting (18):

$$\left|\Big(M(\tilde{\nu}(t)) - M(\tilde{\nu}(0))\Big)\cdot\big(\tilde{\nu}(t) - \tilde{\nu}(0)\big)\right| = o(t^2)$$

as $t\to 0$. This proves that the limits on the right hand sides in (14) and (20) are equal.

Let us recall a classical geometrical definition: two smooth oriented surfaces S and S' are said to be *parallel at distance* δ if S' is the image of S through the map $x\mapsto x+\delta\vec{n}_S(x)$, where $\vec{n}_S$ is the normal vector field on S. It is classical that the normal vector on S' at $x+\delta\vec{n}_S(x)$ is actually $\vec{n}_S(x)$. (We will give a proof of this result in the next section.) In particular, S is also a surface parallel to S', at distance $-\delta$. Moreover, if S have well defined radii of curvature $\rho_i(x)$ $(i=1,2)$, then S' also have radii of curvature at $x+\delta\vec{n}_S(x)$, equal to $\rho_i(x)+\delta$.

So we see that for a body K of constant width α with median surface M_K, the median surface and the boundary ∂K are parallel at distance $\pm\alpha$, whenever they are smooth. In general, these surfaces are not smooth, but only have Lipschitz regularity, though.

3 Parametrizations

In this section, we give a parametrization of the median surface of a body K of constant width. This provides a simple parametrization of the boundary of K, and gives a simple formula to compute the volume and surface area of K.

From now on we focus on the three-dimensional setting. A similar work can easily be done in dimension two, but the properties of orbiforms are already quite well known.

3.1 Isothermal parametrization of the sphere

Let us start with a parametrization of the unit sphere $\mathbf{S}^2$ in the form $(u,v) \in \Omega \mapsto \nu(u,v)$, where Ω is some subset of $\mathbb{R}^2$. We assume that this parametrization is *isothermal*, that is, satisfies for all $(u,v) \in \Omega$:

$$\partial_u\nu(u,v)\cdot\partial_v\nu(u,v) = 0 \qquad \text{and} \qquad |\partial_u\nu(u,v)| = |\partial_v\nu(u,v)| =: \frac{1}{\lambda(u,v)}. \tag{21}$$

We also assume that the map $\nu : \Omega \to \mathbf{S}^2$ is injective and almost surjective, that is, its image set is equal to $\mathbf{S}^2$ except possibly a finite number of points.

An example of such a parametrization is

$$(u,v) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \longmapsto \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, \tanh v\right) \tag{22}$$

and in such a case $\lambda(u,v) = \cosh v$, and $\nu(\Omega) = \mathbf{S}^2 \setminus \{(0,0,\pm 1)\}$. However we do not rely on this particular form in the following.

For technical reasons, we will also assume that λ satisfies, for all values of (u,v), the identity

$$\lambda^2\nabla\cdot\left(\lambda^{-1}\nabla\lambda\right) = \lambda\Delta\lambda - |\nabla\lambda|^2 = 1. \tag{23}$$

(Gradient and Laplacian taken relative to (u,v).) This is clearly true for the particular parametrization given above.

Let us shorten the notations by not writing the dependencies on the parameters (u,v). We introduce the unit vectors $\nu_u := \lambda\partial_u\nu$, $\nu_v := \lambda\partial_v\nu$. Since ν is also a unit vector, we have $\nu\cdot\partial_u\nu = 0$, so $\nu\cdot\nu_u = 0$; and similarly $\nu\cdot\nu_v = 0$. Hence the family (ν,ν_u,ν_v) is an orthonormal basis of $\mathbb{R}^3$, taking (21) into account.

Lemma 3 *For such an isothermal parametrization of the unit sphere, we have*

$$\partial_u \nu_u = -\lambda^{-1}\nu + \lambda^{-1}\partial_v \lambda\, \nu_v \tag{24}$$
$$\partial_v \nu_u = -\lambda^{-1}\partial_u \lambda\, \nu_v \tag{25}$$
$$\partial_u \nu_v = -\lambda^{-1}\partial_v \lambda\, \nu_u \tag{26}$$
$$\partial_v \nu_v = -\lambda^{-1}\nu + \lambda^{-1}\partial_u \lambda\, \nu_u \tag{27}$$

Proof.

Since $\nu \cdot \partial_u \nu = 0$, we get by differentiating $\nu \cdot \partial^2_{uv}\nu = -\partial_u \nu \cdot \partial_v \nu = 0$, so $\partial^2_{uv}\nu$ has the form $\alpha\nu_u + \beta\nu_v$. On the other hand

$$\partial^2_{uv}\nu = \partial_u(\partial_v \nu) = \partial_u(\lambda^{-1}\nu_v) = \lambda^{-1}\partial_u \nu_v - \lambda^{-2}\partial_u \lambda\, \nu_v.$$

Since $|\nu_v| = 1$ implies $\nu_v \cdot \partial_u \nu_v = 0$, we get $\beta = \partial^2_{uv}\nu \cdot \nu_v = -\lambda^{-2}\partial_u \lambda$. Similarly $\alpha = -\lambda^{-2}\partial_v \lambda$. Putting this relation in the value of $\partial^2_{uv}\nu$ above, we deduce (26). We get (25) using $\partial^2_{uv}\nu = \partial_v(\partial_u \nu)$ in the same way.

Differentiating the three relations $|\nu_u|^2 = 1$, $\nu \cdot \nu_u = 0$ and $\nu_u \cdot \nu_v$ with respect to u, we get $\nu_u \cdot \partial_u \nu_u = 0$,

$$\nu \cdot \partial_u \nu_u = -\nu_u \cdot \partial_u \nu = -\lambda^{-1} |\nu_u|^2 = -\lambda^{-1}$$

and

$$\nu_v \cdot \partial_u \nu_u = -\nu_u \cdot \partial_u \nu_v = \lambda^{-1}\partial_v \lambda.$$

This gives (24). The proof of (27) is similar. $\square$

Let us finish this section with a note about the antipodal symmetry on $\mathbf{S}^2$ that we will use in the following sections. There must be some involutive map $\sigma : \Omega \to \Omega$ such that $\nu \circ \sigma(u, v) = -\nu(u, v)$ for all $(u, v) \in \Omega$. For instance with the parametrization (22) we have

$$\nu(u + \pi, -v) = -\nu(u, v) \tag{28}$$

so $\sigma : (u, v) \mapsto (u + \pi, -v)$. We will call this map the *antipodal symmetry of the parametrization*. In the following, we will always assume that σ is C^1 and is *consistent* with the isothermal parametrization, that is satisfies:

$$\lambda \circ \sigma = \lambda. \tag{29}$$

Since $\nu = -\nu \circ \sigma$, we have $\partial_u \nu = -\partial_u \sigma \partial_u \nu \circ \sigma$. Considering the norm of both sides, we deduce with (29) that $|\partial_u \sigma| = 1$. Similarly we have $|\partial_v \sigma| = 1$. Since $\sigma \in \mathrm{C}^1$, we see that

$$\partial_u \sigma = \text{const.} = \pm 1 \quad \text{and} \quad \partial_u \sigma = \text{const.} = \pm 1. \tag{30}$$

These relations, together with the definition of ν_u, ν_v and (29) imply

$$\nu_u \circ \sigma = -\partial_u \sigma \nu_u, \qquad \nu_v \circ \sigma = -\partial_v \sigma \nu_v. \tag{31}$$

3.2 Parametrization of the median surface

Since the three vectors (ν, ν_u, ν_v) are independent, any point $P \in \mathbb{R}^3$ can be written in the form $P = h\nu + h_1 \nu_u + h_2 \nu_v$. If h, h_1, h_2 are actually some smooth functions of (u, v), P depends on (u, v) and describe a surface. In this section we investigate the conditions on h, h_1, h_2 ensuring that such a surface is the median surface of a spheroform, with support vector $\nu(u, v)$ at $P(u, v)$.

Proposition 2 *Given an isothermal parametrization $\nu : \Omega \to \mathbf{S}^2$ of the sphere, let K be a strictly convex body. There exists a C^1 map $h : \Omega \to \mathbb{R}$ such that $R_K(\nu) = \mathcal{M}(h)(u, v)$ for all $\nu = \nu(u, v)$, where*

$$\mathcal{M}(h) : \left| \begin{array}{rcl} \Omega & \longrightarrow & \mathbb{R}^3 \\ (u, v) & \longmapsto & h\,\nu + \lambda \partial_u h\, \nu_u + \lambda \partial_v h\, \nu_v. \end{array} \right. \tag{32}$$

Proof. For any given $\nu = \nu(u, v)$, consider $P(u, v) := R_K(\nu(u, v))$. Since the three vectors (ν, ν_u, ν_v) are independent, $P(u, v)$ can be written in the form $P = h\nu + h_1 \nu_u + h_2 \nu_v$, for some functions h, h_1, h_2 of (u, v). These functions are continuous since R_K is continuous.

Note that $h_K(\nu(u, v)) = \nu(u, v) \cdot P(u, v) = h(u, v)$. So h is just the support function of K, and in particular is of class C^1. Moreover we have from the definition of R_K:

$$\forall (u_1, v_1) \in \Omega, \qquad P(u_1, v_1) \cdot \nu \leq P \cdot \nu.$$

(All values of the functions are at (u, v), unless otherwise specified.) Let us write this relation with $u_1 = u + t, v_1 = v$. For small values of t, we have from (24–27):

$$\begin{aligned} \nu(u + t, v) &= \nu + t\lambda^{-1} \nu_u + o(t), \\ \nu_u(u + t, v) &= \nu_u - t\lambda^{-1} \big(\nu - \partial_v \lambda \nu_v\big) + o(t), \\ \nu_v(u + t, v) &= \nu_v - t\lambda^{-1} \partial_v \lambda \nu_u + o(t). \end{aligned}$$

Also since h is of class C^1, we have $h(u+t,v) = h + t\partial_u h + o(t)$. Hence

$$\begin{aligned} 0 = P\cdot\nu - h &\geq P(u_1,v_1)\cdot\nu - h \\ &\geq t\Big(\partial_u h - \lambda^{-1}h_1(u+t,v)\Big) + o(t). \end{aligned}$$

Passing to the limit $t = 0$ with either $t > 0$ or $t < 0$, we deduce that $h_1(u,v) = \lambda\partial_u h$. Similarly $h_2 = \lambda\partial_v h$. $\square$

Remark 3.B. Notice that $\mathcal{M}(h)$ is obviously linear with respect to h. Since in the previous proposition, $h = \mathcal{M}(h)\cdot\nu = R_K\cdot\nu$ is the support function of K, then the mapping from K to h is additive (with respect to the Minkowski addition). However, not any h yields an interesting body K. In particular, if $h(u,v) = \vec{w}\cdot\nu(u,v)$ for some fixed vector $\vec{w}\in\mathbb{R}^3$, then $\partial_u h = \vec{w}\cdot\partial_u\nu = \lambda^{-1}\vec{w}\cdot\nu_u$, so $\mathcal{M}(h) = \vec{w}$ is constant, and the corresponding body K reduces to a point. Due to the additivity property, we see that adding $\vec{w}\cdot\nu$ to some given h is equivalent to a translation of the corresponding body K by the vector $\vec{w}$.

We prove in the next theorem that for a constant width body, the corresponding function h is actually $\mathrm{C}^{1,1}$ (the derivatives are lipschitzian). Here and in the following, differential operators like ∇ (gradient) or Δ (laplacian) are taken relative to the variables (u,v). We denote by $\nabla^\perp$ the operator $(-\partial_v,\partial_u)$. Whenever h is twice differentiable, we denote by D^2h the 2×2 matrix of its second-order derivatives (hessian matrix).

An inequality like $D^2h(u,v)\leq A$, where A is also a 2×2 symmetrical matrix, means that the difference $A - D^2h(u,v)$ is nonnegative definite. For $h\in\mathrm{C}^{1,1}$ only, the second-order derivatives do not necessarily exists, but the Taylor expansion

$$T[h](u,v;\xi,\eta) := h(u+\xi,v+\eta) - h(u,v) - \xi\partial_u h(u,v) - \eta\partial_v h(u,v)$$

is of order $O(\xi^2+\eta^2)$ for (ξ,η) small.

Definition 1 *We shall say that* $D^2h(u,v)\leq A = (a_{i,j})$ in a generalized sense*, if the following occurs:*

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{T[h](u,v;\xi,\eta) - \frac{1}{2}\big(a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2\big)}{\xi^2+\eta^2} \leq 0. \tag{33}$$

Similarly we say that $D^2h(u,v)\geq A$ in a generalized sense*, if a similar property holds with a limit-inf* ≥ 0 *instead.*

Clearly this is the same as the usual meaning for a twice-differentiable function h, since

$$T[h](u,v;\xi,\eta) = \frac{1}{2}\xi^2\partial^2_{uu}h(u,v) + \xi\eta\partial^2_{uv}h(u,v) + \frac{1}{2}\eta^2\partial^2_{vv}h(u,v) + o(\xi^2+\eta^2)$$

in that case.

Definition 2 *Given an isothermal parametrization $\nu : \Omega \to \mathbf{S}^2$ of the sphere, let σ be its antipodal symmetry. Let $\mathrm{C}^{1,1}_\sigma(\Omega)$ be the set of all $\mathrm{C}^{1,1}$ maps $h : \Omega \to \mathbb{R}$ such that*

$$h \circ \sigma = -h. \tag{34}$$

Let $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ be the subset of functions $h \in \mathrm{C}^{1,1}_\sigma(\Omega)$ satisfying everywhere on Ω in a generalized sense (see Definition 1 above):

$$-\frac{\alpha}{2\lambda^2}\,\mathrm{Id} \leq U[h] \leq \frac{\alpha}{2\lambda^2}\,\mathrm{Id} \tag{35}$$

where

$$U[h] := D^2h + \lambda^{-2}h\,\mathrm{Id} + \lambda^{-1}\nabla\lambda \otimes \nabla h - \lambda^{-1}\nabla^\perp\lambda \otimes \nabla^\perp h. \tag{36}$$

Theorem 3 *Given an isothermal parametrization of the sphere, let $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ be given by the Definition 2 above.*

Then an application $M : \mathbf{S}^2 \to \mathbb{R}^3$ is the median surface of a spheroform if and only if there exists $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ such that $M(\nu) = \mathcal{M}(h)(u,v)$ for all $\nu = \nu(u,v)$, where the map $\mathcal{M}(h) : \Omega \to \mathbb{R}^3$ is defined by (32). In this case, the map $\mathcal{M}(h + \frac{\alpha}{2}) : \Omega \to \mathbb{R}^3$ describes all but a finite number of the points on ∂K.

The restriction about exceptional points on ∂K comes from the fact that $\nu(\Omega)$ equals $\mathbf{S}^2$, excepts some exceptional points. (The points $(0,0,\pm 1)$ with the parametrization (22).)

Proof. Given $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$, define $M : \mathbf{S}^2 \to \mathbb{R}^3$ by $M(\nu) = \mathcal{M}(h)(u,v)$ for all $\nu = \nu(u,v)$. Let us prove that M is the median surface of some spheroform. In view of Theorem 2, Proposition 1 and Remark 2.A, we just have to prove (5) and (20). From (29–31) we get

$$M(-\nu) = \mathcal{M}(h) \circ \sigma(u,v) = -h \circ \sigma\nu + \lambda\partial_u h \circ \sigma\partial_u\sigma\nu_u + \lambda\partial_v h \circ \sigma\partial_v\sigma\nu_v.$$

But (34) implies in particular $\partial_u h \circ \sigma = \partial_u h / \partial_u\sigma$ and a similar relation for v. So $\mathcal{M}(h) \circ \sigma = \mathcal{M}(h)$ and M satisfies (5).

Let us now prove (20). Any $\widetilde{\nu} \in \mathrm{C}^2([0,1];\mathbf{S}^2)$ can be written in the parametrization as $\widetilde{\nu}(t) = \nu(u(t),v(t))$ where $u(t), v(t) \in \mathrm{C}^2([0,1])$. If $\widetilde{\nu}(0) =$

$\nu(u_0,v_0) =: \nu_0$, we also have $u(0)=u_0$, $v(0)=v_0$. Let us consider $\xi := u(t)-u_0$, $\eta := v(t)-v_0$. Since $\partial_u\nu = \lambda^{-1}\nu_u$, and

$$\partial^2_{uu}\nu = \partial_u\big(\lambda^{-1}\nu_u\big) = \lambda^{-2}\big(-\nu+\partial_v\lambda\nu_v-\partial_u\lambda\nu_u\big)$$

with the help of (24). With similar relations for the other derivatives, we get the Taylor expansion of $\widetilde{\nu}$ near $t=0$:

$$\begin{aligned}\widetilde{\nu}(t) &= \nu(u_0+\xi,v_0+\eta) = \nu_0+\xi\lambda^{-1}\nu_u+\eta\lambda^{-1}\nu_v\\ &+\frac{1}{2\lambda^2}\Big[\xi^2(-\nu+\partial_v\lambda\nu_v-\partial_u\lambda\nu_u)-2\xi\eta(\partial_u\lambda\nu_v+\partial_v\lambda\nu_u)+\eta^2(-\nu+\partial_u\lambda\nu_u-\partial_v\lambda\nu_v)\Big]\\ &\qquad\qquad+o(\xi^2+\eta^2), \quad (37)\end{aligned}$$

where all functions on the right hand side are computed at (u_0,v_0).

In particular we get using $\xi = u(t)-u_0 = t\dot u(0)+o(t)$ and $\eta = v(t)-v_0 = t\dot v(0)+o(t)$,

$$\dot{\widetilde{\nu}}(0) = \lim_{t\to 0}\frac{1}{t}(\widetilde{\nu}(t)-\nu_0) = \lambda^{-1}(\dot u(0)\nu_u+\dot v(0)\nu_v).$$

This implies

$$\frac{1}{\lambda^2}(\xi^2+\eta^2) = t^2\left|\dot{\widetilde{\nu}}(0)\right|^2+o(t^2). \tag{38}$$

Similarly since $M(\nu(u,v))\cdot\nu(u,v) = h(u,v)$ from the definition of h, we have:

$$\begin{aligned}&\big(M(\widetilde{\nu}(t))-M(\nu_0)\big)\cdot\widetilde{\nu}(t)\\ &\quad= h(u(t),v(t))-M(\nu_0)\cdot\nu(u(t),v(t))\\ &\quad= h(u_0+\xi,v_0+\eta)-(h\nu_0+\lambda\partial_u h\nu_u+\lambda\partial_v h\nu_v)\cdot\nu(u_0+\xi,v_0+\eta)\\ &\quad= h(u_0+\xi,v_0+\eta)-h-\xi\partial_u h-\eta\partial_v h+\frac{h}{2\lambda^2}(\xi^2+\eta^2)\\ &\qquad+\frac{1}{2\lambda}\Big(\xi^2(-\partial_v\lambda\partial_v h+\partial_u\lambda\partial_u h)+2\xi\eta(\partial_u\lambda\partial_v h+\partial_v\lambda\partial_u h)\\ &\qquad\qquad+\eta^2(-\partial_u\lambda\partial_u h+\partial_v\lambda\partial_v h)\Big)\\ &\qquad+o(\xi^2+\eta^2)\\ &\quad= T[h](u_0,v_0;\xi,\eta)+\frac{1}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}A\begin{pmatrix}\xi & \eta\end{pmatrix}+o(t^2)\end{aligned}$$

where $A := \lambda^{-2}h\,\mathrm{Id}+\lambda^{-1}\nabla\lambda\otimes\nabla h-\lambda^{-1}\nabla^\perp\lambda\otimes\nabla^\perp h$.

Using the right inequality in (35), and the definition of the corresponding generalized sense, we deduce with (38):

$$\limsup_{t\to 0}\frac{1}{t^2}\big(M(\widetilde{\nu}(t))-M(\nu_0)\big)\cdot\widetilde{\nu}(t) \le \limsup_{t\to 0}\frac{\alpha}{4\lambda^2t^2}(\xi^2+\eta^2) = \frac{\alpha}{4}\left|\dot{\widetilde{\nu}}(0)\right|.$$

Similarly the left inequality in (35) yields the reverse inequality, which achieves the proof of (20).

Conversely, let K be a spheroform. We know from Proposition 2 that there exists some function $\tilde{h} \in \mathrm{C}^1(\Omega)$ such that $R_K(\nu) = \mathcal{M}(\tilde{h})(u,v)$ for all $\nu = \nu(u,v)$.

Consider now the function $h := \tilde{h} - \frac{\alpha}{2}$. From the definition of $\mathcal{M}$, it is clear that $\mathcal{M}(h)(u,v) = \mathcal{M}(\tilde{h}) - \frac{\alpha}{2}\nu(u,v)$, so for any $\nu = \nu(u,v)$ we have

$$M_K(\nu(u,v)) = R_K(\nu(u,v)) - \frac{\alpha}{2}\nu(u,v) = \mathcal{M}(h)(u,v).$$

Moreover the map $\nu \mapsto M_K(\nu)$ is lipschitzian from Lemma 2. Hence $\partial_u h(u,v) = M_K(\nu) \cdot \nu_u(u,v)$ is lipschitzian, too. And similarly for $\partial_v h$. So $h \in \mathrm{C}^{1,1}$. Additionally $h(u,v) = M_K(\nu(u,v)) \cdot \nu(u,v)$ implies $h \circ \sigma = M_K(-\nu) \cdot (-\nu) = -h$, so h satisfies (34). Hence $h \in \mathrm{C}^{1,1}_\sigma(\Omega)$.

We know that M_K satisfies (20). If we consider the special path $\tilde{\nu}$: $t \mapsto \nu(u_0 + t\xi, v_0 + t\eta)$, we can expand $\widetilde{\nu}(t)$ near $t = 0$ as before, obtaining something similar to (37). This implies with a similar computation:

$$\big(M(\widetilde{\nu}(t) - M(\nu_0)\big) \cdot \widetilde{\nu}(t) = t^2 T[h](u_0, v_0; \xi, \eta) + \frac{t^2}{2}\begin{pmatrix}\xi \\ \eta\end{pmatrix} A \begin{pmatrix}\xi & \eta\end{pmatrix} + o(t^2).$$

Therefore (20) implies (35) in the generalized sense. This completes the proof that $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$. $\square$

3.3 Regularity of the parametrization

In this section, we investigate the consequences of (35) on h, whenever h is regular enough.

Proposition 3 *Let h be C^2 on some open set $\omega \subset \Omega$. Then h satisfies* (35) *on ω if and only if it satisfies*

$$|R(h)| \le \min\left(\frac{\alpha}{\lambda}, \frac{\alpha}{2\lambda} + \frac{2\lambda}{\alpha}J(h)\right) \tag{39}$$

on ω, where $R(h)$ and $J(h)$ are the trace and determinant of the matrix $\lambda^{-1}hI + \nabla\lambda \otimes \nabla h - \nabla^\perp\lambda \otimes \nabla^\perp h + \lambda D^2 h$, that is

$$R(h) := \frac{2h}{\lambda} + \lambda\Delta h \tag{40}$$

$$\begin{aligned} J(h) := {} & \lambda^{-2}h^2 + h\Delta h + \lambda^2 \det D^2 h + \lambda\nabla^\perp\lambda \cdot D^2 h \cdot \nabla^\perp h \\ & - \lambda\nabla\lambda \cdot D^2 h \cdot \nabla h - |\nabla\lambda|^2\ |\nabla h|^2 . \end{aligned} \tag{41}$$

Proof. For a C^2 function, the generalized sense for (35) is just the common pointwise sense. We can multiply by λ and get (39) since a 2×2 matrix is nonnegative definite, if, and only if, its trace and determinant are nonnegative. $\square$

Let us note for further references that $R(h)$ and $J(h)$ are the trace and determinant of a symmetric matrix. Therefore it has real eigenvalues, and in particular the discriminant of its characteristic polynomial is nonnegative:

$$R(h)^2 \geq 4J(h). \tag{42}$$

This holds for any C^2 function h.

Notice that for any $\delta \in \mathbb{R}$,

$$R(h+\delta) = R(h) + \frac{2\delta}{\lambda}, \tag{43}$$

$$J(h+\delta) = J(h) + \delta\lambda^{-1}R(h) + \lambda^{-2}\delta^2. \tag{44}$$

Therefore (39) may be equivalently written

$$R(h+\frac{\alpha}{2}) \geq 0, \quad R(h-\frac{\alpha}{2}) \leq 0, \quad J(h+\frac{\alpha}{2}) \geq 0 \quad \text{and} \quad J(h-\frac{\alpha}{2}) \geq 0. \tag{45}$$

Remark 3.C. The appearance of the matrix in the previous proposition seems quite odd at first. Here is another way to obtain it, which is easier to understand, but requires again $h \in C^2$, so we can compute the derivatives of $M := \mathcal{M}(h)$. We get using (24–27):

$$\partial_u M = a\nu_u + b\nu_v \quad \text{and} \quad \partial_v M = c\nu_u + d\nu_v \tag{46}$$

where

$$\begin{aligned} a &:= \lambda^{-1}h + \partial_u\lambda\partial_u h - \partial_v\lambda\partial_v h + \lambda\partial^2_{uu}h \\ b = c &:= \partial_v\lambda\partial_u h + \partial_u\lambda\partial_v h + \lambda\partial^2_{uv}h \\ d &:= \lambda^{-1}h + \partial_v\lambda\partial_v h - \partial_u\lambda\partial_u h + \lambda\partial^2_{vv}h. \end{aligned}$$

So we find $DM\,\nu = 0$ in agreement to Proposition 1. We also see from their definition that $R(h) = a + d$ and $J(h) = ad - bc$.

Since $M = \mathcal{M}(h) \in C^1$, (14) is equivalent to (15) according to Proposition 1. Since for $\nu_0 = \nu(u_0, v_0)$, we have $\nu_0^{\perp} = \mathrm{Span}(\nu_u(u_0, v_0), \nu_v(u_0, v_0))$, we just have to check (15) for $w = \xi\nu_u + \eta\nu_v$, with arbitrary (ξ, η). This inequality becomes then, using (24–27):

$$\forall(\xi,\eta) \in \mathbb{R}^2, \quad \left|a\xi^2 + 2b\xi\eta + d\eta^2\right| \leq \frac{\alpha}{2\lambda}(\xi^2+\eta^2).$$

This means

$$-\frac{\alpha}{2\lambda}\,\mathrm{Id} \leq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leq \frac{\alpha}{2\lambda}\,\mathrm{Id}$$

in the sense of matrices, which is (35).

Remark 3.D. The previous proposition has also a geometrical meaning and can be proved using corresponding considerations. Indeed, in matrix notations, we have $\nabla M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \nu_u \\ \nu_v \end{pmatrix}$, using again the notations of the previous remark. Consequently we get:

$$\nabla \nu = \begin{pmatrix} \partial_u \nu \\ \partial_v \nu \end{pmatrix} = \lambda^{-1} \begin{pmatrix} \nu_u \\ \nu_v \end{pmatrix} = \lambda^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \nabla M.$$

By definition, the curvatures of the surface $(u, v) \mapsto M(u, v)$ are the eigenvalues of the matrix A such that $\nabla\nu = A\nabla M$, since ν is normal to the surface. And the curvature radii, their inverse, are the eigenvalues of A^{-1}. We see that in our case $A = \lambda^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$. So the curvature radii are the solutions ρ_i $(i = 1, 2)$ of the equation

$$\rho^2 - \lambda^2 R(h)\rho + \lambda^2 J(h) = 0. \tag{47}$$

Therefore, if we change h to $\tilde{h} := h + \delta$ in order to consider a parallel surface, we see that the curvature radii $\tilde{\rho}_i$ on this new surface are solutions of the equation

$$\begin{aligned} 0 &= \tilde{\rho}^2 - (2\delta + 2h + \lambda^2\Delta h)\tilde{\rho} + \lambda^2 J(h) + \delta(2h + \lambda^2\Delta h) + \delta^2 \\ &= (\tilde{\rho} + \delta)^2 - (2h + \lambda^2\Delta h)(\tilde{\rho} + \delta) + \lambda^2 J(h). \end{aligned}$$

Hence $\tilde{\rho}_i = \rho_i + \delta$ as claimed before.

For a body of constant width α, the parallel to the median surface at distance $\pm\frac{\alpha}{2}$ are part of the boundary of K. So they are convex, with opposite directions (the outward normal vector on $\mathcal{M}(h-\frac{\alpha}{2})(u, v)$ is $-\nu(u, v)$). Hence we must have $\rho_i \in [-\frac{\alpha}{2}, +\frac{\alpha}{2}]$. This is equivalent to saying that the left hand side of (47) is nonnegative whenever $\rho = \pm\frac{\alpha}{2}$, and that the sum of the roots belongs to $[-\alpha, \alpha]$. This in turn is equivalent to (39). In other words, (39) expresses the fact that the radii of curvature on the median surface are in $[-\frac{\alpha}{2}, +\frac{\alpha}{2}]$, whenever they are defined.

The equivalent formula (45) expresses the fact that the Gaussian curvatures $J(h \pm \frac{\alpha}{2})$ are nonnegative, while the mean curvatures $R(h \pm \frac{\alpha}{2})$ have opposite signs, since the convex surfaces are opposite.

3.4 Surface area and volume

According to Theorem 3, there is a one to one correspondence between $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ and $\mathcal{W}_\alpha$. We investigate now the way to compute the volume and surface area of some $K \in \mathcal{W}_\alpha$ through the corresponding function h.

Proposition 4 *Let $\omega \subset \Omega$ be a symmetrical subset of the parametrization space, that is $\sigma(\omega) = \omega$. Let $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ be C^2 on $\partial\omega$, and let K be the corresponding spheroform.*

The set $R_K(\omega) \subset \partial K$ has surface area:

$$\begin{aligned} |R_K(\omega)| = \frac{\alpha^2}{4}\,|\nu(\omega)| + \int_\omega \left(\lambda^{-2}h^2 - \frac{1}{2}\,|\nabla h|^2\right) \\ + \int_{\partial\omega}\left(h + \frac{1}{2}\lambda^2\Delta h\right)\nabla h\cdot\vec{n} - \frac{1}{4}\int_{\partial\omega}\nabla\left(\lambda^2\,|\nabla h|^2\right)\cdot\vec{n}. \end{aligned} \tag{48}$$

(Here $|\nu(\omega)|$ stands for the surface area of the subset $\nu(\omega)$ of $\mathbf{S}^2$.)

Proof. It is enough to prove the proposition for $h \in \mathrm{C}^2(\overline{\omega})$. Indeed an approximation argument allows to generalize to others h, since the right hand side of (48) involves second-order derivatives only on the boundary of ω.

We make use of the notations of Proposition 3 and Remark 3.C. We have

$$\partial_u M \times \partial_v M = (a\nu_u + b\nu_v) \times (c\nu_u + d\nu_v) = J(h)\nu.$$

Hence the area of the surface $\mathcal{M}(h)(\omega)$ is $\int_\omega |J(h)|$. Since $\mathcal{M}(h)$ is the median surface of $K \in \mathcal{W}_\alpha$, $\mathcal{M}(h + \frac{\alpha}{2})$ and $\mathcal{M}(h - \frac{\alpha}{2})$ both describe the boundary of K. If we restrict the parameters to the subset ω, they both describe $R_K(\omega)$ since we assumed $\omega = \sigma(\omega)$. So the surface area of $R_K(\omega)$ is equal to $\int_\omega \left|J(h \pm \frac{\alpha}{2})\right|$. This implies, using (44) and (45):

$$\begin{aligned} |R_K(\omega)| &= \frac{1}{2}\int_\omega J(h + \frac{\alpha}{2})(u,v)\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{2}\int_\omega J(h - \frac{\alpha}{2})(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_\omega \left(\frac{\alpha^2}{4\lambda^2(u,v)} + J(h)(u,v)\right)\mathrm{d}u\,\mathrm{d}v. \\ &= \frac{\alpha^2}{4}\,|\nu(\omega)| + \int_\omega J(h)\,\mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

(The latter equality follows from $\partial_u\nu \times \partial_v\nu = \lambda^{-2}\nu$.) To complete the proof of the proposition, we have now to prove:

$$\begin{aligned} \int_\omega J(h)(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_\omega \left(\lambda^{-2}h^2 - \frac{1}{2}\,|\nabla h|^2\right) \\ + \int_{\partial\omega}\left(h + \frac{1}{2}\lambda^2\Delta h\right)\nabla h\cdot\vec{n} - \frac{1}{4}\int_{\partial\omega}\nabla\left(\lambda^2\,|\nabla h|^2\right)\cdot\vec{n}. \end{aligned} \tag{49}$$

By expanding products in (41), we get

$$J(h) = \lambda^{-2}h^2 + h\Delta h + J_1(h) - J_2(h), \tag{50}$$

where

$$J_1(h) := \lambda^2 \left(\partial^2_{uu}h\partial^2_{vv}h - (\partial^2_{uv}h)^2\right) \\ + \lambda \left(\partial_u\lambda\partial_u h\partial^2_{vv}h + \partial_v\lambda\partial_v h\partial^2_{uu}h - \partial_u\lambda\partial_v h\partial^2_{uv}h - \partial_v\lambda\partial_u h\partial^2_{uv}h\right)$$

and

$$J_2(h) := \left((\partial_u\lambda)^2 + (\partial_v\lambda)^2\right)\left((\partial_u h)^2 + (\partial_v h)^2\right) \\ + \lambda \left(\partial_u\lambda\partial_u h\partial^2_{uu}h + \partial_v\lambda\partial_v h\partial^2_{vv}h + \partial_v\lambda\partial_u h\partial^2_{uv}h + \partial_u\lambda\partial_v h\partial^2_{uv}h\right).$$

Let us define $w_1 := \partial_u h\partial^2_{uv}h - \partial_v h\partial^2_{uu}h$, $w_2 := \partial_u h\partial^2_{vv}h - \partial_v h\partial^2_{uv}h$ and $\vec{w} := (w_1, w_2)$. We have $\partial_u w_2 - \partial_v w_1 = 2(\partial^2_{uu}h\partial^2_{vv}h - (\partial^2_{uv}h)^2)$. Therefore

$$2J_1(h) = \partial_u\left(\lambda^2 w_2\right) - \partial_v\left(\lambda^2 w_1\right).$$

This implies using Green's formula

$$\int_\omega J_1(h) = \frac{1}{2}\int_{\partial\omega}\lambda^2\vec{w}\cdot\mathrm{d}\vec{\ell}.$$

Now let us denote by H the scalar function $|\nabla h|^2$. Since $\vec{w} = \Delta h\nabla^\perp h - \frac{1}{2}\nabla^\perp H$ (where $\nabla^\perp = (-\partial_v, \partial_u)$), we also have

$$\int_\omega J_1(h) = \frac{1}{2}\int_{\partial\omega}\lambda^2\left(\Delta h\nabla h - \frac{1}{2}\nabla H\right)\cdot\vec{n}\,\mathrm{d}s.$$

Considering now J_2, we can check easily that $J_2(h) = |\nabla\lambda|^2 H + \frac{1}{2}\lambda\nabla\lambda\cdot\nabla H$. Integrating by parts we get

$$\begin{aligned}\int_\omega J_2(h) &= \int_\omega H\left(|\nabla\lambda|^2 - \frac{1}{2}\nabla\cdot(\lambda\nabla\lambda)\right) + \frac{1}{2}\int_{\partial\omega}H\lambda\nabla\lambda\cdot\vec{n}\,\mathrm{d}s \\ &= \frac{1}{2}\int_\omega H\left(|\nabla\lambda|^2 - \lambda\Delta\lambda\right) + \frac{1}{2}\int_{\partial\omega}H\lambda\nabla\lambda\cdot\vec{n}\,\mathrm{d}s \\ &= -\frac{1}{2}\int_\omega H + \frac{1}{2}\int_{\partial\omega}H\lambda\nabla\lambda\cdot\vec{n}\,\mathrm{d}s\end{aligned}$$

using (23).

Finally we have

$$\int_\omega h\Delta h = \int_{\partial\omega}h\,|\nabla h|\cdot\vec{n}\,\mathrm{d}s - \int_\omega|\nabla h|^2 = \int_{\partial\omega}h\nabla h\cdot\vec{n}\,\mathrm{d}s - \int_\omega H.$$

So integrating (50) yields (49). $\square$

We are now in position to compute the volume and surface area of any spheroform, expressed as integrals of the corresponding function h:

Theorem 4 *Let $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ be given, and $K \in \mathcal{W}_\alpha$ the corresponding spheroform. The surface area $|\partial K|$ and the volume $|K|$ are given by*

$$|\partial K| = \int_\Omega \left(\lambda^{-2}h^2 - \frac{1}{2}|\nabla h|^2\right) + \pi\alpha^2, \tag{51}$$

$$|K| = \frac{\alpha}{2}\int_\Omega \left(\lambda^{-2}h^2 - \frac{1}{2}|\nabla h|^2\right) + \frac{\pi\alpha^3}{6}. \tag{52}$$

Corollary 5 (Blaschke) *Let K be any convex body of constant width α in dimension 3. Then the volume and surface area of K satisfy:*

$$|K| = \frac{\alpha}{2}|\partial K| - \frac{\pi\alpha^3}{3}. \tag{53}$$

We refer the reader to [1] for the original proof of this property. Here it follows directly from Theorem 4.

Proof. The parametrization domain $\Omega = \mathbf{S}^1 \times \mathbb{R}$ has no boundary. So if we apply (48) with $\omega = \Omega$, we get (51) since $|\nu(\Omega)| = |\mathbf{S}^2| = 4\pi$.

The volume of K can be expressed as $|K| = \frac{1}{3}\int_{\partial K} \overrightarrow{OM} \cdot \vec{n}\,\mathrm{d}\sigma$, using Stokes' formula. We can choose $M = \mathcal{M}(h + \frac{\alpha}{2})(u,v)$ as a parametrization, and then $\vec{n} = \nu(u,v)$ and $\mathrm{d}\sigma = J(h+\frac{\alpha}{2})\,\mathrm{d}u\,\mathrm{d}v$. But we may also choose $M = \mathcal{M}(h - \frac{\alpha}{2})$, and in such a case $\vec{n} = -\nu(u,v)$ since $\vec{n}$ is the outward normal in Stokes' formula, and $\mathrm{d}\sigma = J(h - \frac{\alpha}{2})\,\mathrm{d}u\,\mathrm{d}v$. So we have

$$|K| = \frac{1}{3}\int_\Omega (h+\frac{\alpha}{2})J(h+\frac{\alpha}{2}) = -\frac{1}{3}\int_\Omega (h-\frac{\alpha}{2})J(h-\frac{\alpha}{2}).$$

In particular this implies, using (44) and an integration by parts:

$$\begin{aligned}
|K| &= \frac{1}{6}\int_\Omega \left\{(h+\frac{\alpha}{2})J(h+\frac{\alpha}{2}) - (h-\frac{\alpha}{2})J(h-\frac{\alpha}{2})\right\}. \\
&= \frac{\alpha}{6}\int_\Omega J(h) + \frac{\alpha^3}{24}\int_\Omega \lambda^{-2} + \frac{\alpha}{6}\int_\Omega h(2\lambda^{-2}h + \Delta h) \\
&= \frac{\alpha}{6}\int_\Omega J(h) + \frac{\pi\alpha^3}{6} + \frac{\alpha}{6}\int_\Omega \left(2\lambda^{-2}h^2 - |\nabla h|^2\right).
\end{aligned}$$

This proves (52) using (49) with $\omega = \Omega$. $\square$

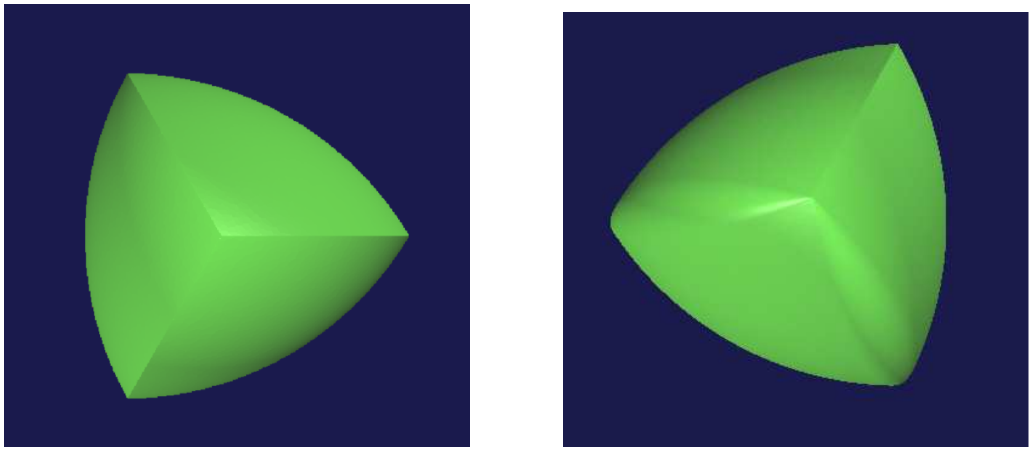

Figure 1: Two views of one Meissner's tetrahedron

3.5 Description of Meissner's tetrahedron

A description of this volume can be found in [3],[15] and [8]. We shall give a brief definition of this volume and describe its parametrization.

Meissner's tetrahedron is geometrically defined in the following way: consider a body K_t obtained as the intersection of four balls of radius α which centers are the vertices of a regular tetrahedron (of edge lengths α). Thus, the boundary of K_t is composed of four pieces of balls connected by six arc of circles. Surprisingly, this set K_t is not of constant width: geometrical considerations show that opposite circular edges are too far away. Meissner proposed to smooth three edges of K_t in order to get a constant width body. Consider E the union of three circular edges which share a common vertex S. Then, the body K defined as

$$K = \bigcap_{x \in E} B(x, \alpha) \cap K_t$$

is a body of constant width called Meissner's tetrahedron (see figure 3.4). Notice that it is possible to build an other constant width body based on the regular etrahedron by smoothing a different set of edges.

We give below an analytical representation in terms of its h function based on the parametrization of the sphere described by (22). In order to take benefit of the invariance of the previous body K by rotations of angles $\pm 2\pi/3$, we consider a body K built on a regular tetrahedron which has its vertex S on the z-axes and the others on the plane $z = 0$. Moreover, we assume that the equilateral triangle formed by other vertices on $z = 0$ is symmetric with respect to the y-axes. It is straightforward to check that

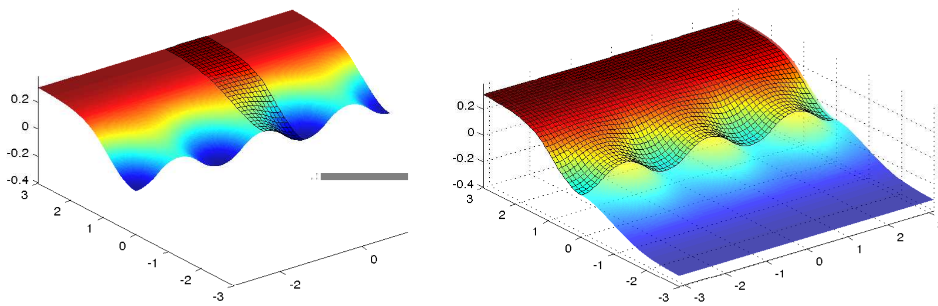

Figure 2: Construction of Meissner's h function

such a Meissner's tetrahedron is invariant with respect to the rotations about the z-axes of angles $\pm 2\pi/3$ and also invariant by orthogonal symmetry with respect to the plane $x = 0$. Then, the function h is completely defined if we give an analytical representation of h on $\omega = [0, \frac{\pi}{3}] \times [0, +\infty[$ since relations (34) and $h(u, v) = h(-u, v)$ define h on all $\Omega = [-\pi, \pi] \times]-\infty, +\infty[$ (see figure 3.5). On ω, for $\alpha = 1$, the function h may be described in the following way:

$$\begin{cases} \sqrt{2/3}\tanh v - 1/2, & \text{if} \sinh v > 2\sqrt{2}\cos u, \\ -1/2 - (1/2\sqrt{3})(\cos u/\cosh v) + \dots & \\ (\sqrt{3}/2)(\cosh v^2 - \sin u^2)/\cosh v, & \text{if} \sinh v \le 2\sqrt{2}\cos u, \cosh v \ge 2\sin u, \\ 1/2 + (1/\sqrt{3})\cos(u + 2\pi/3)/\cosh v & \text{if} \sinh v \le 2\sqrt{2}\cos u, \cosh v > 2\sin u. \end{cases}$$

Notice that it is possible to compute the volume and the surface area of Meissner's tetrahedron thanks to equations (51) and (53). After some symbolic computations, we get the formulas presented in [7]:

$$|K| = \frac{2\pi}{3} - \frac{\pi\sqrt{3}}{4}\arccos\frac{1}{3}$$

$$|\partial K| = 2\pi - \frac{\pi\sqrt{3}}{2}\arccos\frac{1}{3}$$

3.6 Local optimality

We now come back to the volume functional $K \mapsto |K|$ in order to investigate the properties of its minimizers. A striking consequence of Theorem 4 is that minimizing the volume in $\mathcal{W}_\alpha$ is equivalent to minimizing the surface area. More precisely, the volume minimization problem is equivalent to

$$\min_{h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)} L(h) \qquad \text{where} \qquad L(h) = \int_\Omega \left(\lambda^{-2} h^2 - \frac{1}{2} |\nabla h|^2 \right). \qquad (54)$$

Let us first observe that the maximum value of L is zero:

Lemma 4 *For any $h \in \mathrm{C}^{1,1}_{\sigma}(\Omega)$, we have $L(h) \leq 0$.*

In particular, a maximizer of L in $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ is always $h = 0$, which corresponds to a ball of radius $\alpha/2$. Hence such a ball has maximal volume among all spheroforms, a well-known result.

Proof. Let W_σ be the space of all functions $h \in W^{1,2}(\Omega)$ satisfying (34). This is a closed subspace of the Sobolev space $W^{1,2}(\Omega)$, so it is a Banach space. Let us define $s \in \mathbb{R}$ as follows:

$$s = \inf_{h \in W_\sigma} \frac{\displaystyle\int_\Omega |\nabla h|^2}{\displaystyle\int_\Omega \lambda^{-2} h^2}.$$

This is a "weighted Sobolev constant", and it is classical in PDE theory that the infimum is actually attained by a smooth function $\phi \in W_\sigma$ satisfying the corresponding Euler equation

$$\int_\Omega \nabla h \cdot \nabla \phi = s \int_\Omega \lambda^{-2} h \phi, \qquad \forall h \in W_\sigma.$$

In other words, ϕ is an eigenfunction of the operator $-\lambda^2 \Delta$, with the symmetry condition $\phi \circ \sigma = -\phi$. Additionally, if we choose two open sets $\Omega_+ \subset \Omega$ and $\Omega_- = \sigma(\Omega_+)$ such that $\Omega_+ \cap \Omega_- = \emptyset$ and $\Omega = \overline{\Omega}_+ \cup \Omega_-$, then it follows from Krein-Rutman's theorem that there exists an eigenfunction ϕ satisfying $\phi > 0$ on Ω_+. One way to choose such a set Ω_+ is to consider some fixed vector $\vec{w} \in \mathbb{R}^3$, and to set

$$\Omega_+ := \big\{(u,v) \in \Omega \,;\, \vec{w} \cdot \nu(u,v) > 0\big\}.$$

Given such a $\vec{w}$, define $g := (u,v) \mapsto \vec{w} \cdot \nu(u,v)$. As explained in Remark 3.B, $\mathcal{M}(g) = \vec{w}$ for all (u,v), and the body corresponding to $h + g$, for any $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$, is just a translation of the body corresponding to h. In particular, they have the same volume, so $L(h+g) = L(h)$. Since L is quadratic, this means that

$$0 = L(g) + \int_\Omega \lambda^{-2} h g - \nabla h \cdot \nabla g$$

for all $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$. In particular, $L(g) = 0$ since we can take $h = 0$, and $\Delta g + 2\lambda^{-2} g$ is orthogonal (for the L^2 scalar product) to all $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$. The latter implies it is orthogonal to W_σ, since $\bigcup_{\alpha>0} \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ contains $\mathrm{C}^2_\sigma(\Omega)$. Hence it is orthogonal to ϕ, so we get

$$2\int_\Omega \lambda^{-2} g\phi = \int_\Omega \nabla g \cdot \nabla \phi = s\int_\Omega \lambda^{-2} g\phi.$$

Now both functions g and ϕ are positive on Ω_+ and odd with respect to σ, so

$$\int_\Omega \lambda^{-2} g\phi = 2\int_{\Omega_+} \lambda^{-2} g\phi > 0$$

and therefore $s = 2$. This implies $L(h) \leq 0$ for any $h \in W_\sigma$, and in particular in $\mathrm{C}^{1,1}_\sigma(\Omega)$.

$\square$

Remark 3.E. Since balls are the unique maximizers of the volume among spheroforms of given width, it follows that for $h \in \mathrm{C}^{1,1}_\sigma(\Omega)$:

$$L(h) = 0 \iff \exists \vec{w} \in \mathbb{R}^3,\ h(u,v) = \vec{w} \cdot \nu(u,v),$$

for all $(u,v) \in \Omega$.

An interesting consequence of the previous lemma is that the functional L is actually strictly concave with respect to h (when considered on the quotient of $\mathrm{C}^{1,1}_\sigma(\Omega)$ by the smallest subspace containing all the functions $\vec{w} \cdot \nu(u,v)$ for $\vec{w} \in \mathbb{R}^3$).

Indeed L is quadratic, so for any $h, g \in \mathrm{C}^{1,1}_\sigma(\Omega)$ and for all $t \in [0,1]$:

$$L(th + (1-t)g) - tL(h) - (1-t)L(g) = -t(1-t)L(h-g) \geq 0.$$

From the remark 3.E, the equality holds if and only if it exists $\vec{w} \in \mathbb{R}^3$ such that $h = g + \vec{w} \cdot \nu(u,v)$.

The following weak optimality result applies not only to global minimizers, but also to local ones. Notice that this condition is very close from the one established in (**??**) for a relaxed problem of (3).

Theorem 6 *Let K be a body of constant width, and a local minimizer of the volume functional. Then K is everywhere irregular in the following sense: for any $A \subset \mathbf{S}^{n-1}$, one of the two subsets $R_K(A)$ or $R_K(-A)$ of ∂K is not a smooth surface.*

In this context, a "smooth surface" means that the set of points can be described as the graph of a regular function. Observe that this result is obvious in dimension two for global minimizers, since these are Reuleaux triangles.

Proof. Let K be a local minimizer of the volume and $A \subset \mathbf{S}^2$ with $R_K(A)$ a smooth surface. Let h be the function of $\mathrm{C}^{1,1}_\sigma(\Omega)$ associated to K by the proposition 2. Since every constant width bodies are strictly convex, we can assume without loss of generality that $R_K(A)$ is the graph of a strictly convex function. In this context, it is standard that the reverse Gauss map is a smooth diffeomorphism. Moreover, the function h is also locally smooth on the points of $\omega \subset \Omega$ corresponding to A since:

$$h(u,v) = R_K(\nu(u,v)) \cdot \nu(u,v).$$

Let us first establish that h saturates the pointwise constraint (35) on a subset of ω. By reducing ω to a smaller set if necessary, we suppose that $\omega \cap \sigma(\omega) = \emptyset$. Assume by contradiction that the four inequalities are strict. Let $g \in \mathrm{C}^2(\omega)$ with compact support. We extend it to $\sigma(\omega)$ by symmetry, defining $g(\sigma(u,v)) = -g(u,v)$, so that the new function, still denoted g, belongs to $\mathrm{C}^{1,1}_\sigma(\Omega)$. Due to the non-saturation property, the functions $f_+ := h + tg$ and $f_- := h - tg$ belong to $\mathrm{C}^{1,1}_\sigma(\Omega)$ for $|t|$ small enough. Now L is strictly concave so we have:

$$L(h) = L(\frac{1}{2}f_+ + \frac{1}{2}f_-) \geq \min(L(f_+), L(f_-)). \tag{55}$$

for all g. Since an equality in (55) is not possible because of the remark 3.E (none of the function $\vec{w} \cdot \nu(u,v)$ has a compact support), we have that $L(h) > \min(L(f_+), L(f_-))$. This contradicts the local minimality of h.

We established in subsection 3.3, that the saturation of the constraints for the regular function h is equivalent to the fact that one or both of the radii of curvature on $R_K(A)$ are equal to α or 0. Since $R_K(A)$ is a strictly convex regular surface, its curvature radii are not zero. As a consequence, on all points of $R_K(A)$ at least one of the curvature radii is equal to α. Consider now the surjective application from ω to $R_K(-A)$ given by

$$(u,v) \mapsto R_K(\nu(u,v)) - \alpha\nu(u,v).$$

If this application is not injective, $R_K(-A)$ is not smooth since at least one point of this surface has a non empty subdiferrential. We conclude that the previous application is an admissible parametrization of $R_K(-A)$. It is now straightforward to compute that on all points of $R_K(-A)$, at least one of the curvature radii is equal to 0. Again, this fact contradicts the regularity of $R_K(-A)$ which concludes the proof. $\square$

Remark 3.F. If we assume additionally that the lines of curvature on $R_K(A)$ of the body K have no torsion, it is possible to show that $R_K(-A)$ is a convex curve. In this situation we would conclude that one of the two pieces of the boundary $R_K(A)$ or $R_K(-A)$ has measure 0. Notice that Meissner's tetrahedron satisfies the previous assumption.

References

[1] Blaschke...

[2] T. Bonnesen, W. Fenchel, *Theory of convex bodies*, BCS Associates, 1987, p. 135-149.

[3] G. D. Chakerian, H. Groemer, *Convexity and its Applications*, Chap. Convex bodies of constant width, ed P. M. Gruber and J. M. Wills, Birkhauser, 1983, p. 49-96.

[4] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[5] J. Dugundji & A. Granas, *Fixed point theory*, Monog. Mat. Polska Ak. **61**, 1982.

[6] E. M. Harrell, *A direct proof ot a theorem of Blaschke and Lebesgue*, Journal of Geometric Analysis, vol 12 (2002), p.81 88.

[7] E. M. Harrell, *Calculations for Convex Bodies. Example: The rotated Reuleaux triangle*, http://www.mathphysics.com/convex/Links/Convexnb_lnk_67.html.

[8] D. Hilbert, *Elf Eigenschaften der Kugel*.

[9] J. B. Hiriart-Urruty, C. Lemaréchal, *Fundamentals of convex analysis*, Springer-Verlag, 2001.

[10] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Series of Comprehensive Studies in Mathematics, Vol. 224, 2. Edition, Springer 1977, 1983.

[11] T. Lachand-Robert, É. Oudet, *Bodies of constant width in arbitrary dimension*, to appear in Math. Nachrichten.

[12] R. T. Rockafellar, *Convex Analysis*, Princeton University Press (1970).

[13] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge Univ. Press, 1993.

[14] K. Toyama, *Self-Parallel Constant Mean Curvature Surfaces*

[15] I.M. Yaglom, V.G. Boltyanskii, *Convex Figures*, Holt, Rinehart and Winston New York.