Shape optimisation under width constraint

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Abstract

In this paper, we present numerical methods to solve optimization problems among convex bodies which satisfy some width constraints. We propose two different numerical methods to handle width equality and width inequality constraints. To illustrate the efficiency of our method, we use our approach to approximate optimal solution of Meissner"s problem and Heil's conjecture.

Keywords: *Optimisation, Convex functions, Numerical schemes, constant width, Reuleaux triangle, Meissner's tetrahedron.*

AMS classification: 46N10, 52A40, 52A41. 52A05, 52A43.

1 Introduction

This article deals with numerical shape optimisation problems involving convex shapes under width constraints in $\mathbb{R}^3$. Throughout the article, we make use of the following notations:

- K is a convex body of $\mathbb{R}^3$ with nonempty interior which contains the origin,
- ∂K denotes its boundary,
- ν_K is the almost everywhere defined outer normal vector field on ∂K, with values on the sphere $\mathbf{S}^2$,
- for $\nu \in \mathbf{S}^2$, $\varphi_K(\nu)$ is the distance to the origin of the supporting plane to K of exterior normal ν. More explicitly,
$$\varphi_K(\nu) = \sup_{x \in K} x.\nu$$
where $x.\nu$ stands for the usual scalar product of $\mathbb{R}^3$. φ_K is called the support function of K,
- $w_K(\nu) = \varphi_K(\nu) + \varphi_K(-\nu)$ for $\nu \in \mathbf{S}^2$ is called the *width* in the direction ν.

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The two kinds of optimisation problem that we will study are :

$$\min_{K \in \mathcal{K}} F(K)$$

where

$$\mathcal{K} = \{K \,\text{convex},\ w_K(\nu) = 1,\ \forall \nu \in \mathbf{S}^2\}. \tag{1}$$

pb_cw

or

$$\mathcal{K} = \{K \,\text{convex},\ w_K(\nu) \geq 1,\ \forall \nu \in \mathbf{S}^2\} \tag{2}$$

pb_minw

In particular, we focus our work on the numerical study of the previous problems when $F(K)$ has a geometrical meaning. More precisely, we restrict our study to $F(K)$ equal to the volume of K denoted by $|K|$ or the surface area of ∂K denoted by S_K.

Taking $F(K)$ equal to $|K|$ (or equivalently to S_K by Blaschke's formula), problem (1) is a well known question called Meissner's conjecture. In dimension two, this problem was solved by Lebesgue and Blaschke: the solution turns out to be a *Reuleaux triangle*. In dimension three, this problem is still open. Indeed the mere existence of non trivial three-dimensional bodies of constant width is not so easy to establish. In particular, no finite intersection of balls has constant width (except balls themselves), a striking difference with the two-dimensional case. A simple construction, to obtain constant width bodies in dimension 3, is to consider a two dimensional body of constant width having an axis of symmetry (like the Reuleaux triangle for instance): the corresponding body of revolution obtained by rotation around this axis is of constant width. F. Meissner proved that the *rotated Reuleaux triangle* has the smaller volume among constant width bodies of revolution. Later on he was able to construct another spheroform (usually called "Meissner's tetrahedron") which does not have the symmetry of revolution. The volume of this body is smaller than any other known of constant width, so it is a good candidate as a solution to the problem (1) (see [9], [10], [11], [12]).

In a first part of this article, we study constant width constraints of type (1) using an analytical parametrisation introduced in [1]. We discuss a cubic spline method based on [6] which approximates problem (1) by a standard quadratic programming problem under equalities and inequalities constraints. The main interest of the method is that it gives a discrete way to parameterise (and not to approximate) constant with bodies. This point is of dramatic importance to study Meissner's conjecture in a numerical way. Based on that method, we perform numerical experiments to study the local optimality of Meissner's body and of the *rotated Reuleaux triangle*. Our numerical results satisfy the weak optimality condition which has been described in [1]. More precisely, any constant width body K^* which minimises the area is irregular in the sense that for any $\omega \in \mathbf{S}^2$ small enough, the part of ∂K^* whose normals are ω and the other part whose normals are $-\omega$ are not both regular.

In a second part we study the relaxed problem (2). The question to minimise the surface area among convex bodies of prescribed minimal width was first addressed in [5]. One convex body based on a regular simplex, whose precise description is recalled in the following, has been conjectured by E. Heil to be optimal in 1978. The previous analytical parametrisation is not relevant in that context. Thus, we give a new algebraic discretisation of convex bodies based on Minkowski's sums. To illustrate the efficiency of our method for inequality constraints, we solve numerically Heil's problem. Our numerical optimisation gives a polytope which is admissible (in the sense that it satisfies exactly the constraints up to round off errors) and has a surface area smaller than Heil's polytope. This result disprove Heil's conjecture.

2 A geometrical approach and its difficulties

For every $\nu \in \mathbf{S}^2$ and every $\varphi \geq 0$, let us define the half-space of $\mathbb{R}^3$:

$$[\![\nu, \varphi]\!] = \left\{x \in \mathbb{R}^3,\ x \cdot \nu \leq \varphi\right\}.$$

In a previous article [7] the authors present a discretisation of convex bodies based on half spaces. A convex set K is approached by a polytope P which is defined in the following way. Let $n \in \mathbb{N}^*$, choose randomly and uniformly n vectors ν_i of $\mathbf{S}^2$ and define

$$P_n = \bigcap_{i=1}^{n} [\![\nu_i, \varphi_i]\!],$$

where $\varphi_i = \varphi_K(\nu_i)$. It is straightforward to show that when n tends to infinity this outer approximation converges with respect to the Hausdorff distance to the set K. This discretisation has been used in [7] to solve numerically different optimisation's problems where convex bodies are involved. The key idea is to start with a given convex polytope and to adjust the parameters φ_i in order to minimise the cost functional.

As it has been noticed in the introduction, a width constraint can be written in terms of the support function. Namely, $w_K(\nu) = 1$ is equivalent by definition to $\varphi_K(\nu)+\varphi_K(-\nu) = 1$. A simple idea would be to reproduce the method of [7] adding linear constraints to the parameters φ_i such that

$$\varphi_i^+ + \varphi_i^- = 1,$$

where $\varphi_i^\pm$ are the parameters associated to the normal vectors $\pm\nu_i$. Here is the crucial difficulty: the latter statement on the parameters $\varphi_i^\pm$ is not equivalent to $\varphi_K(\nu_i)+\varphi_K(-\nu_i) = 1$. It may happen for instance that

$$\bigcap_{i=1}^{n-1} [\![\nu_i, \varphi_i]\!] \subset \{x \cdot \nu_n \leq \varphi_n - \varepsilon\}$$

with $\varepsilon > 0$. In this case the hyperplane

$$\left\{x \in \mathbb{R}^3,\ x \cdot \nu_n = \varphi_n\right\}$$

is not anymore in a tangent position since it has an empty intersection with the body P_n. This difficulty turns the previous algorithm inefficient for this kind of constraint.

We present in this article two alternative methods to handle width constraints in geometrical optimisation. Those two discretisations of problems (1) and (2) leads to standard non-convex quadratic programming problems which are solved by classical solvers (see section 3.3.1).

3 Minimisation among sets of constant width

In this section we are interested in the numerical study of Meissner's conjecture. Does Meissner's tetrahedron minimise the volume (or equivalently in dimension 3, the surface area) among sets of constant and fixed width (see [1] for a complete description of this convex body) ? In order to be able to eventually contradict the conjecture we have to propose a discrete description of constant width bodies which is an exact sub-problem of (1). More precisely, we would like to restrict our optimisation procedure to a subset of $\mathcal{K}$. Moreover, we would like to be able to evaluate exactly (up to round off errors) its surface area in order to compare our results and Meissner's conjecture.

We first recall a functional parametrisation result of constant width bodies obtained in [1]. Based on this parametrisation, problem (1) becomes a more classical optimisation problem on some convex space functional. Then, in order to approximate an optimal function we follow an approach introduced in [6] based on tensor-product splines. We stress the point on the fact that our method gives at the end of the process a discrete description (based on the cubic splines parametrisation) of some real constant width body of $\mathcal{K}$. Based on the previous formulation, our optimisation problem becomes a large scale quadratic optimisation problem. Finally, some numerical results are presented.

3.1 Parametrisation by the median surface

A major difficulty to handle the constant width constraint is the potential irregularity of those bodies. As it is suggested by the 2 dimensional case, we have to consider shapes which may have singularities (consider for instance Reuleaux's triangle which solves the question we are interested in, in dimension 2).

A framework designed to parametrise those kind of potentially irregular shapes is presented in [1]. We recall here the main results related to this parametrisation which will be useful to describe our optimisation approach.

First, we recall from [1] that constant width sets can all be described by vector fields on the sphere which satisfy the following global conditions :

lian surface **Theorem 1** *Let $\alpha > 0$ be given and $M : \mathbf{S}^2 \to \mathbb{R}^3$ be a continuous application satisfying*

$$\forall \nu \in \mathbf{S}^2, \qquad M(-\nu) = M(\nu); \tag{3}$$

eq-M even

$$\forall \nu_0, \nu_1 \in \mathbf{S}^2, \qquad \big(M(\nu_1) - M(\nu_0)\big) \cdot \nu_0 \leq \frac{\alpha}{4} \left|\nu_1 - \nu_0\right|^2. \tag{4}$$

eq-M varia

Define a subset $K \subset \mathbb{R}^n$ as follows:

$$K := \Big\{ M(\nu) + t\nu \,;\, \nu \in \mathbf{S}^2, t \in \Big[0, \frac{\alpha}{2}\Big] \Big\}. \tag{5}$$

eq-def K p

Then K is a convex body of constant width α.

Conversely, any convex body of constant width α can be described by (5) *with some vector field M satisfying* (3) *and* (4).

Next, we recall that the previous vector fields M on $\mathbf{S}^2$ can be parametrised by some smooth scalar functions satisfying second order differential conditions.

To this purpose, consider a parametrisation of the sphere $(u,v) \in \Omega \mapsto \nu(u,v) \in \mathbf{S}^2$, where Ω is some subset of $\mathbb{R}^2$. We assume that this parametrisation is *isothermal*, that is, satisfies for all $(u,v) \in \Omega$:

$$\partial_u \nu(u,v) \cdot \partial_v \nu(u,v) = 0 \quad \text{and} \quad |\partial_u \nu(u,v)| = |\partial_v \nu(u,v)| =: \frac{1}{\lambda(u,v)}. \tag{6}$$

eq-isoth-s

Let K a body of constant width, then there exists a C^1 map $h : \Omega \to \mathbb{R}$ such that

$$M(\nu) = \mathcal{M}(\nu(u,v)) = h\,\nu + \lambda \partial_u h\ \nu_u + \lambda \partial_v h\ \nu_v \tag{7}$$

eq-parame

for all $(u,v) \in \Omega$, where M is a vector field associated to K defined by theorem 1.

Conversely, sets of constant width are all described analytically with additional constraints on the previous function h. In order to present those conditions, we recall the two definitions:

generalized

Definition 1 *We shall say that $D^2 h(u,v) \le A = (a_{i,j})$ in a generalised sense, if the following occurs:*

$$\limsup_{(\xi,\eta)\to(0,0)} \frac{T[h](u,v;\xi,\eta) - \frac{1}{2}\big(a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2\big)}{\xi^2+\eta^2} \le 0. \tag{8}$$

eq-D2h ge

where

$$T[h](u,v;\xi,\eta) := h(u+\xi, v+\eta) - h(u,v) - \xi \partial_u h(u,v) - \eta \partial_v h(u,v)$$

Similarly we say that $D^2 h(u,v) \ge A$ in a generalised sense, if a similar property holds with a limit-inf ≥ 0 instead.

Notice that in the regular case (that is h of class C^2), the inequality $D^2 h(u,v) \le A$ is equivalent to the standard positiveness of the matrix $A - D^2 h(u,v)$.

efi-espace C

Definition 2 *Given an isothermal parametrisation $\nu : \Omega \to \mathbf{S}^2$ of the sphere, let σ be its antipodal symmetry. Let $\mathrm{C}^{1,1}_\sigma(\Omega)$ be the set of all $\mathrm{C}^{1,1}$ maps $h : \Omega \to \mathbb{R}$ such that*

$$h \circ \sigma = -h. \tag{9}$$

eq-h odd

Let $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ be the subset of functions $h \in \mathrm{C}^{1,1}_\sigma(\Omega)$ satisfying everywhere on Ω in a generalised sense (see Definition 1 above):

$$-\frac{\alpha}{2\lambda^2}\,\mathrm{Id} \le U[h] \le \frac{\alpha}{2\lambda^2}\,\mathrm{Id} \tag{10}$$

eq-cond h

where

$$U[h] := D^2 h + \lambda^{-2} h\ \mathrm{Id} + \lambda^{-1} \nabla\lambda \otimes \nabla h - \lambda^{-1} \nabla^\perp \lambda \otimes \nabla^\perp h. \tag{11}$$

eq-def de

We can now recall the main result obtain in [1] to describe constant width bodies. Then we have the characterisation of constant width set in terms of their support function:

form with h **Theorem 2** *Given an isothermal parametrisation of the sphere, let $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ be given by the Definition 2 above. Then an application $M : \mathbf{S}^2 \to \mathbb{R}^3$ is the median surface of a constant width body (that is corresponds to a constant width body by 5) if and only if there exists $h \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$ such that $M(\nu) = \mathcal{M}(h)(u,v)$ for all $\nu = \nu(u,v)$, where the map $\mathcal{M}(h) : \Omega \to \mathbb{R}^3$ is defined by (7). In this case, the map $\mathcal{M}(h + \frac{\alpha}{2}) : \Omega \to \mathbb{R}^3$ describes all but a finite number of the points on ∂K.*

3.2 Discretisation of the $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$

Based on theorem 2, the discretisation of our optimisation problem can be reduced to the discretisation of the space functional $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$. We follow an approach introduced in [6] based on tensor-product splines to obtain splines which satisfy exactly (and not approximately) the differential constraints (10).

In the following we will use the standard isothermal parametrisation of the sphere ν:

$$(u,v) \in \Omega \longmapsto \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, \tanh v \right) \tag{12}$$ eq-param s

where $\Omega = [-\pi, \pi] \times \mathbb{R}$, $\lambda(u,v) = \cosh v$, and $\nu(\Omega) = \mathbf{S}^2 \setminus \{(0,0,\pm 1)\}$.

The starting point of our approach is to discretise the space of parameters $[-\pi, \pi] \times [0, v_{max}]$ by a bounded regular orthogonal grid where v_{max} is a parameter of the method. In order to satisfy exactly the antipodal symmetry constraint (9), we impose to the grid to contain the origin. Consider now a tensor-product spline h_d defined on that grid. We want to find sufficient conditions on the coefficients of h_d which ensure that $h_d \in \mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$. Since the final goal of the discretisation is to achieve an optimisation procedure, we want the constraints on the coefficients of h_d to be linear.

Notice first that the periodicity and the antipodal symmetry constraint (9) are equivalent to linear equality constraints on those coefficients. The most challenging problem is to manage the constraints (10) in a linear way. We will describe how to deduce a set of linear inequality constraints which is asymptotically equivalent to those conditions. For simplicity we restrict our description to the differential inequality

$$0 \le U[h_d] + \frac{\alpha}{2\lambda^2}\,\mathrm{Id}. \tag{13}$$ constr_cwl

In [6], the author describes how to obtain a set of linear inequality which ensures that the tensor-product spline to be a convex function. In that sense it is an interior approximation of the convexity constraint. Moreover, it is also proved that any strictly convex patch satisfies this kind of constraints for a suitable choice of the set of constraints. Due to the weight λ which appears in (10), we need to adapt the method to the space $\mathrm{C}^{1,1}_{\sigma,\alpha}(\Omega)$. Let us first describe more precisely the constraint (13) on a patch of the tensor-product spline assuming for simplicity $\alpha = 1$:

$$\begin{pmatrix} \partial_{uu} h_d + \dfrac{h_d + 1/2}{\lambda^2} - \dfrac{\sinh(v)\partial_v h_d}{\lambda} & \partial_{uv} h_d + \dfrac{\sinh(v)\partial_u h_d}{\lambda} \\ \partial_{uv} h_d + \dfrac{\sinh(v)\partial_u h_d}{\lambda} & \partial_{vv} h_d + \dfrac{h_d + 1/2}{\lambda^2} + \dfrac{\sinh(v)\partial_v h_d}{\lambda} \end{pmatrix} \geq 0 \tag{14}$$

The key point is to remark that the previous matrix may be rewritten only in terms of $\tanh(v)$ and $\tanh^2(v)$ by the standard formula $1/\lambda^2 = 1 - \tanh^2(v)$. Regarding $Y := \tanh(v)$ as a new parameter and using the approach of [6], we can force the differential constraints constr_cwbi3 by a set of linear inequalities imposed on the coefficients of the cubic spline. We do not recall here all the technical description of those inequalities but we illustrate the principle of the method in the following to avoid a continuous set of linear constraints depending on Y. If one consider one of the inequalities provided by [6] regarding Y as a parameter, it is straightforward to observe that is has the form

$$Y^2 l_1 + Y l_2 + l_3 \leq 0 \tag{15}$$

linearc

where l_1, l_2 and l_3 are affine forms of the Bernstein/Bezier coefficients of the cubic polynomial h_d on the patch which is considered. Notice that we want (15) to be satisfied for all $Y \in [\tanh(v_1), \tanh(v_2)]$ for some $v_1 < v_2$ depending on the patch. In order to reduce this set of constraints to a finite number of inequalities we use the same strategy as in [6]. Consider the polynomial of two variables

$$p(x, y) = xyl_1 + \frac{x+y}{2} l_2 + l_3. \tag{16}$$

Let $\Sigma = (\sigma_0 \ldots, \sigma_Q)$ be a strictly increasing sequence satisfying

$$\sigma_0 = v_1 < \cdots < \sigma_Q = v_2$$

for some integer $Q > 1$. We define a new set of inequalities

$$\mathcal{I}(l_1, l_2, l_3) = \{\, p(v_1, v_1) \leq 0,\ p(v_2, v_2) \leq 0, \text{ and } p(\sigma_i, \sigma_{i+1}) \leq 0 \ \forall i = 0 \ldots Q-1 \}\,. \tag{17}$$

ineqfin

Following the proof of the Lemma 1 of [6] we obtain:

Lemma 1 *Let $(l_1, l_2, l_3) \in \mathbb{R}^3$, $v_{max} > 0$ and $Q \in \mathbb{N}^*$. Suppose that (l_1, l_2, l_3) satisfies a set of constraints of type (17) for some increasing sequence*

$$\sigma_0 = v_1 < \cdots < \sigma_Q = v_2.$$

Then (l_1, l_2, l_3) satisfies (15) for all $Y \in [v_1, v_2]$.

Proof. First observe that $p(\sigma_i, \sigma_i) \leq 0, \ \forall i = 0 \ldots Q$. If $i = 0, Q$ this is a consequence of the definition of $\mathcal{I}(l_1, l_2, l_3)$. For $0 < i < Q$, we have

$$p(\sigma_i, \sigma_i) = \frac{\sigma_{i+1} - \sigma_i}{\sigma_{i+1} - \sigma_{i-1}} p(\sigma_i, \sigma_{i-1}) + \frac{\sigma_i - \sigma_{i-1}}{\sigma_{i+1} - \sigma_{i-1}} p(\sigma_i, \sigma_{i+1}) \leq 0$$

since both coefficients are positive. Now let $Y \in [v_1, v_2]$, it exists i such that $Y \in [\sigma_i, \sigma_{i+1}]$. In the same way as before we have

$$\begin{aligned} p(Y,Y) &= p(\frac{Y-\sigma_i}{\sigma_{i+1}-\sigma_i}\sigma_{i+1} + \frac{\sigma_{i+1}-Y}{\sigma_{i+1}-\sigma_i}\sigma_i, Y) \\ &= \frac{Y-\sigma_i}{\sigma_{i+1}-\sigma_i}p(\sigma_{i+1},Y) + \frac{\sigma_{i+1}-Y}{\sigma_{i+1}-\sigma_i}p(\sigma_i,Y) \\ &= \left(\frac{Y-\sigma_i}{\sigma_{i+1}-\sigma_i}\right)^2 p(\sigma_{i+1},\sigma_{i+1}) + \left(\frac{\sigma_{i+1}-Y}{\sigma_{i+1}-\sigma_i}\right)^2 p(\sigma_i,\sigma_i) \\ &\quad +2\frac{(Y-\sigma_i)(\sigma_{i+1}-Y)}{(\sigma_{i+1}-\sigma_i)^2}p(\sigma_{i+1},\sigma_i). \end{aligned} \tag{18}$$

Since $p(Y,Y)$ is equal to $Y^2 l_1 + Y l_2 + l_3$ by definition, the inequality follows for all $Y \in [v_1, v_2]$.

□

By this lemma we are able, up to the introduction of the new parameter Σ, to describe a set of linear constraints on the coefficients of the cubic spline which ensure that the body associated to h_d by (7) is of constant width.

To conclude the description of our optimisation approach, we recall from [1] that the surface area $|\partial K|$ of a body of constant width defined by its support function h can be evaluated by the formula:

$$|\partial K| = \int_\Omega \left(\lambda^{-2}h^2 - \frac{1}{2}|\nabla h|^2\right) + \pi\alpha^2, \tag{19}$$ eq-area bo

By the last equality, the surface area associated to the constant width body defined by h_d is a quadratic form of its Berstein/Bezier coefficients. This observation complete the description of our internal approximation of problem (1) as a large scale quadratic problem. We describe below the numerical optimal conditions which have been used to solve that quadratic problem.

3.3 Numerical results

All the discrete constraints that have been considered up to now are the discretisation of local constraints. This matter of fact has a crucial impact on the complexity of the discrete optimisation problem since we have to deal only with sparse constraints.

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3.3.1 Computer implementation

In order to take benefit of this sparsity we used the efficient large scale optimisation software LANCELOT of the GALAHAD library developed by N. Gould D.Orban and P. Toint (see [4] and [3]).

Let us describe more precisely the local optimality conditions that the LANCELOT module tries to reach. As explained in [3], a general nonlinear constrained optimisation problem can be reformulated in the form:

$$\min_{x\in\mathbb{R}^N} f(x)$$

where x is subject to the equality constraints

$$c_j(x) = 0 \quad 1 \leq j \leq m,$$

and the simple bounds

$$l_i \leq x_i \leq u_i, \quad 1 \leq j \leq N.$$

The algorithm implemented in LANCELOT is based on an Augmented Lagrangian Method. At each step, an approximate minimiser of the augmented Lagrangian function

$$\Phi(x, \lambda, S, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i c_i(x) + \frac{1}{2\nu} \sum_{i=1}^{m} s_{ii} c_i(x)^2$$

is found (the parameter ν and the factors s_{ii} are adjusted by the program). Let P be the projection operator on the bound constraints, namely:

$$P(x, l, u)_i = \begin{cases} l_i & \text{if} \qquad x_i < l_i \\ u_i & \text{if} \qquad x_i > u_i \\ x_i & \text{otherwise.} \end{cases}$$

The algorithm stops when the two conditions

$$||x - P(x - \nabla_x L(x, \lambda), l, u)||_\infty \leq \varepsilon_l \tag{20}$$

and

$$||c(x)||_\infty \leq \varepsilon_c \tag{21}$$

where ε_l and ε_c are precision factors which are prescribed by the user.

3.3.2 Our results

We present in the following figures the results of our approach. The first point we are interested in is to check the local optimality of Meissner's tetrahedron. We then compute analytically the h function (see [1] for the complete expression) which defines this body and project h on the grid we are working on. This set of values is the starting point of our numerical process. We present in figure 1 the starting h function and Meissner's body.

Nb of variables (with the gap variables)	Nb of active bounds	Constraints in $\|\|.\|\|_\infty$	Projected gradient in $\|\|.\|\|_\infty$
3772	965	2.6509E-04	6.8437E-04
5967	1692	7.8426E-04	7.8741E-04
8662	2495	9.6875E-04	5.3467E-04 table-newton

Table 1: Precision obtained with the grids 41x20, 51x25, 61x30 table:tab

Actually, it has not been possible to distinguish the initial shape and the result produced by the optimisation: Meissner's body is, at least in a numerical way, a local minimiser of

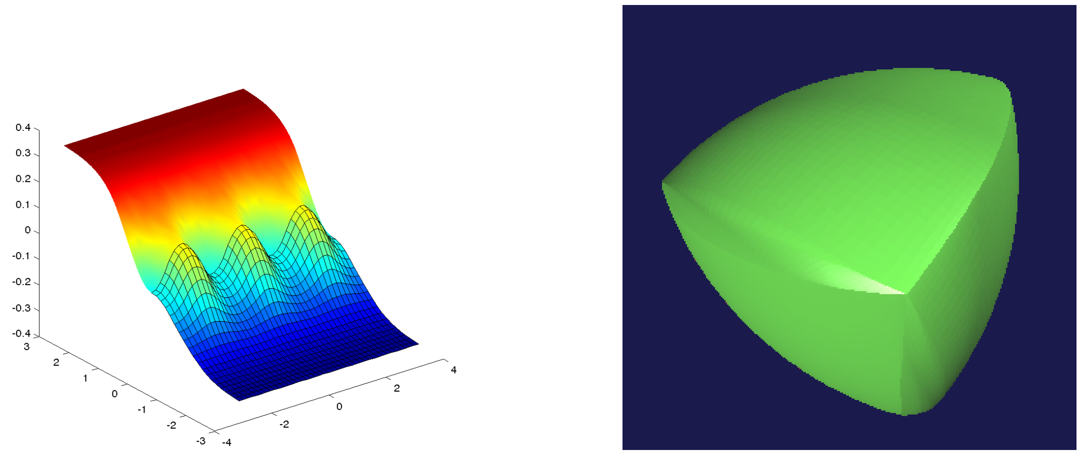

Figure 1: Meissner's h function and the associated body

meissner_

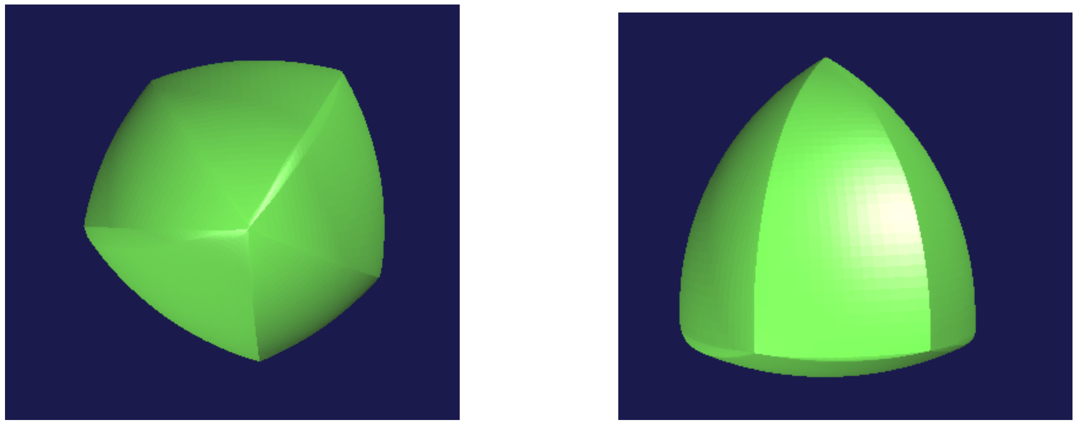

Figure 2: Result of the local optimisation of reuleaux rotated body

result_re

the surface area among constant width bodies. We give in table 1 numerical details of the precision reached with Meissner's tetrahedron as initial guess on different grids.

The same experiment has been carried out starting from the Reuleaux's rotated triangle. This body is known to be the body of least surface area among constant width body of revolution. By this experiment we wanted to study the optimality of that body in the larger class of sets of constant width. As it is reported in figures 2 and 3, this body is not numerically speaking a local optima: the critical shape that has been found seems to be build on a Reuleaux pentagon by the process described in [8]. To conclude, as reported in the introduction, notice that the shape of figure 2 satisfies the weak optimality condition which has been described in [1]: for any $\omega \in \mathbf{S}^2$ small enough, the part of the boundary of that body which normals are ω and the opposite part which normals are $-\omega$ are not both regular.

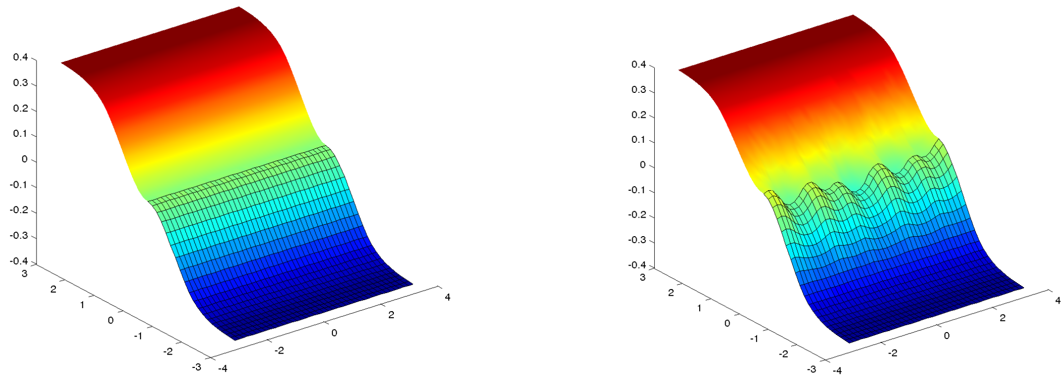

Figure 3: Initial and final h support functions h_reulrota

4 Minkowski sums: an algebraic discretisation for inequality constraints

In the following we are interested in the approximation of an optimal solution of the following problem:

$$\min_{K \in \mathcal{K}} S_K, \tag{22}$$ pb_relaxe

where $\mathcal{K} = \{K \subset \mathbb{R}^3, \quad \text{convex}, \quad w_K(\nu) \geq 1, \quad \forall \nu \in \mathbf{S}^2\}$ and S_K stands for the surface area of the body K. Before introducing our approach, let us first recall some basic facts on Minkowski's sum of two sets $A, B \subset \mathbb{R}^3$. We define Minkowski's sum of sets A and B by

$$A + B = \{x + y, (x, y) \in A \times B\}.$$

An interesting feature related to the width of a convex set and Minkowski's sum is its almost linear behaviour. Let $\lambda, \mu \in \mathbb{R}_+^*$, A and B two convex sets of $\mathbb{R}^N$, then $\lambda A + \mu B$ is convex and its support function is given by

$$\varphi_{\lambda A + \mu B} = \lambda \varphi_A + \mu \varphi_B. \tag{23}$$ eq1

When A and B are subsets of $\mathbb{R}^3$ with nonempty interior, the surface area of the resulting body $S_{\lambda A + \mu B}$ is deduced by the formula

$$S_{\lambda A + \mu B} = \lambda^2 S_A + \mu^2 S_B + \lambda \mu (S_{A+B} - S_A - S_B). \tag{24}$$ eq2

We refer to [13] or [2] for the proof of the previous equality and many other results on convex bodies.

4.1 Outline of the algorithm

Equations (23) and (24) are the starting points of our first method. Let $(K_i)_{i \in I}$ be a finite family of convex sets of $\mathbb{R}^3$ which contain the origin. Consider the approximation of $\mathcal{K}$ obtained by the cone

$$\mathcal{C}_I = \left\{ \sum_{i \in I} \lambda_i K_i, \lambda_i \in \mathbb{R}^+ \right\} \tag{25}$$ coneKi

where the positive vector $\lambda = (\lambda_1, \dots)$ is restricted to the subset of vectors which satisfy $\sum_{i\in I} \lambda_i K_i \in \mathcal{K}$. By relation (23), the constraint $\sum_{i\in I} \lambda_i K_i \in \mathcal{K}$ is equivalent to impose inequality constraints depending on polytopes (K_j) to the coefficients (λ_j). That is

$$\sum_{i\in I} \lambda_i \varphi_{K_i}(\nu) \geq 1 \;\forall \nu \in \mathbf{S}^2 \tag{26}$$

equ3

It is classical that the convex polytope $\sum_{i\in I} \lambda_i K_i$ may have a huge number of vertices. Thus it is not possible to impose exactly the previous constraints. Then, we approximate (26) by a naive discretisation of $\mathbf{S}^2$: let $\nu_1, \dots \nu_m$ be m randomly chosen vectors of the sphere. We consider the finite set of constraints:

$$\varphi_{\sum_i \lambda_i K_i}(\nu_k) + \varphi_{\sum_i \lambda_i K_i}(-\nu_k) \geq 1, \quad k = 1, \dots, m$$

which are equivalent thanks to (23) to

$$\sum_i (b_{ik}^+ + b_{ik}^-)\lambda_i \geq 1, \quad k = 1, \dots, m \tag{27}$$

linconst

where $b_{ik}^{\pm} = \varphi_{K_i}(\pm\nu_k)$. Thus, solutions of the sub-problem

$$\min_{K \in \mathcal{C}_{\mathcal{I}}} S_K, \tag{28}$$

may be approximated by the solutions of the quadratic program:

$$\min_{\lambda} \sum_{i,j} a_{ij}\lambda_i\lambda_j, \tag{29}$$

quadprog1

for vectors λ which satisfy (27). Moreover according to (24), the coefficients a_{ij} can be explicitly estimated by the relations

$$\left\{ \begin{array}{lll} a_{ij} = & \frac{1}{2}(S_{K_i+K_j} - S_{K_i} - S_{K_j}) & i \neq j, \\ a_{ij} = & S_{K_i} & i = j. \end{array} \right.$$

At this step one main difficulty remains. How do we choose the family (K_i) in an effective way in order to get a reasonable approximation of $\mathcal{K}$?

4.2 The cone $\mathcal{C}_I$

4.2.1 The algorithm

Since it is difficult, to estimate numerically quantities like $S_{K_i+K_j}$ when the convex bodies K_i or K_j have a great number of vertices, we would like to be able to approximate a convex body as a Minkowski's sum of "simple" polytopes. Whereas it is true in $\mathbb{R}^2$ that every convex polytope can be decomposed as a finite sum of triangles and segments, the situation is dramatically more complex in dimension 3. Actually, any generic convex polytope (that is a polytope every 2-faces of which are triangles) is indecomposable (see [14]). Thus, if we want to generate a sequence of bodies (K_i) whose associated cone (25) converges to $\mathcal{K}$, we

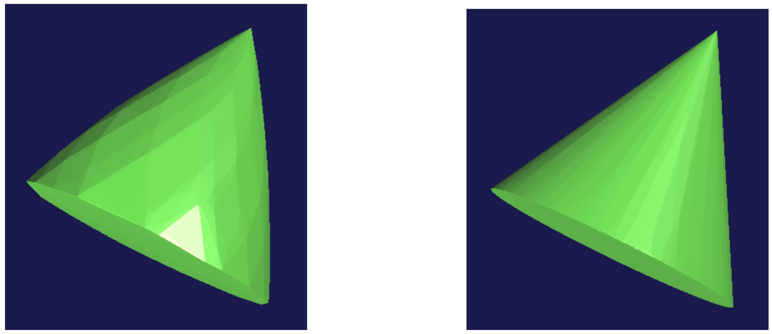

Figure 4: Approximation of a cone by Minkowski sums fig:cone

do not have to restrict ourselves to simplices. We propose the following iterative process to handle this difficulty.

Fix l the maximum number of extremal points of an element of the sequence (K_i^0) and n the number of elements of this family. Let $\varepsilon > 0$ be a precision parameter and j_{max} the maximum number of iterations.

0. Set $j = 0$. Choose randomly n convex polytopes (K_i^0) with at most l extremal points.
1. Solve the optimisation problem (29) associated to the family (K_i^j).
2. Let I^j be the subset of indices of the optimal vector λ^0 whose components are greater in absolute value than ε. Construct a new family (K_i^{j+1}) keeping the bodies $(K_i^j)_{i\in I^j}$ and choosing randomly the others.
3. Let $j \leftarrow j + 1$. If $j > j_{max}$ or $I = I^j$ stop, otherwise go to `1.`

Of course the parameter j_{max} has to be adjusted in relation with the CPU time needed for solving the optimisation step `1`. There is no simple way to give convergence estimates with respect to the choice of the parameters n, l and ε. We propose hereafter one simple numerical experiment to adjust those parameters.

4.2.2 Numerical tests

In order to choose relevant values for parameters n, l and ε, we test our discretisation by Minkowski's sum to approximate a truncated cone. The cone seems to us difficult to approximate by sum of random polytopes since its normal directions cover only a subset of $\mathbf{S}^2$ of dimension 1. To measure the quality of the approximation we introduce a cost functional based on the Euclidian distance between support functions. Consider one truncated cone K and its support function φ_K. In order to observe if our algorithm is able to generate a sequence K^j of bodies which converges to K we define the following cost function:

$$D_K(K^j) = \sum_k (\varphi_K(\nu_k) - \varphi_{K^j}(\nu_k))^2,$$

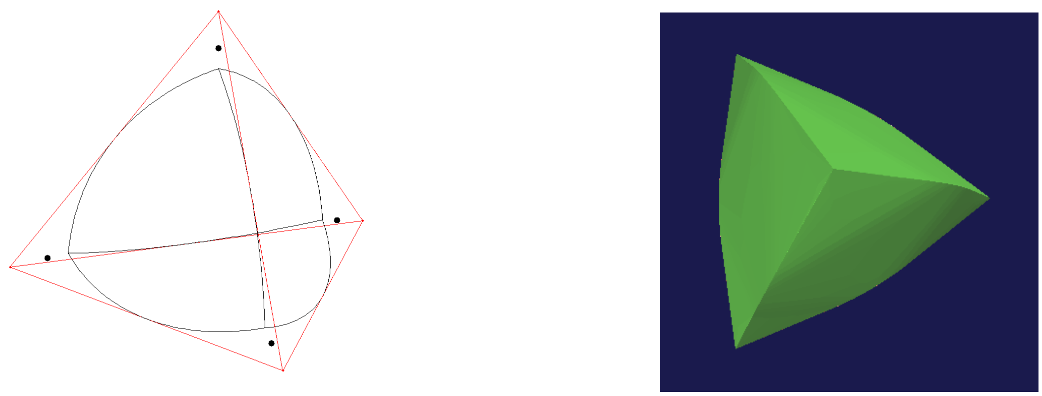

Figure 5: The body of E. Heil heil

where (ν_k) is a fixed list of arbitrary vectors of $\mathbf{S}^2$. Thanks to (23), the auxiliary optimisation problem that we solve at step `1.` is the quadratic problem in λ:

$$\min_{\lambda\geq 0}\sum_k(\varphi_K(\nu_k)-\sum_i \lambda_i\varphi_{K_i^j}(\nu_k))^2.$$

We present in Figures 4 the results we obtained for K equal to a regular cone. The values that have been used to obtain this approximation, are $\#I=100$, $\varepsilon=10^{-6}$, $j_{max}=10^5$, $l=10$ and 10^4 normal vectors ν_k. Notice that a large number of iterations are required in order to get a satisfactory sequence of bodies. This constraint requires an efficient and fast solver for the optimisation step.

4.3 The relaxed problem and the conjecture of E. Heil

In this section we apply the previous method to a more realistic situation which was addressed by E. Heil in [5] p. 261. We look for a solution of

$$\min_{K\in\mathcal{K}} S_K, \tag{30}$$

where $\mathcal{K}=\{K\subset\mathbb{R}^3,\quad \text{convex},\quad w_K(\nu)\geq 1,\quad \forall\nu\in\mathbf{S}^2\}$.

As it has been explained, the latter problem can be approximated by a sequence of quadratic problems. Exactly the same method applied on the problem of minimising the volume would lead to solve a sequence of cubic problems. Up to now, there is no efficient way to solve numerically dense and large cubic problems which makes our method irrelevant in this situation.

E. Heil propose the following construction of its optimal body: Consider the regular tetrahedron of edge-length 1 and replace each edge by a circular arc of radius $\sqrt{2}/2$ and center in the middle of the opposite edge. Take the four points of distance $\sqrt{2}/2$ to the facets of the tetrahedron which are on the line between a vertex and the center of the opposite facet of the tetrahedron. E. Heil claims that the convex hull of the previous 4

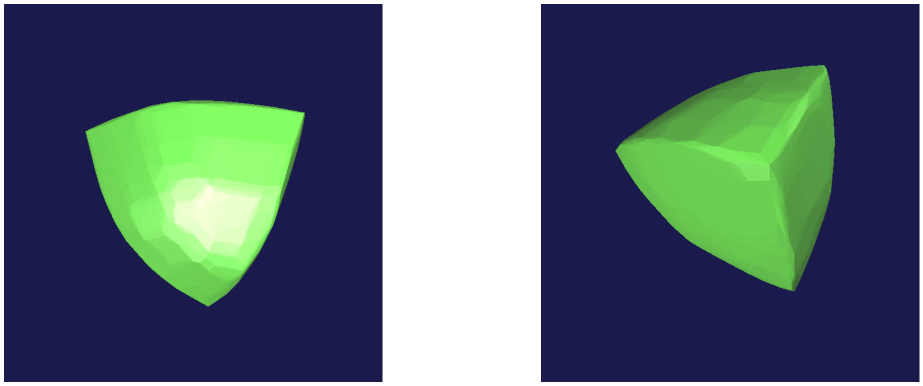

Figure 6: Approximation of the body of fixed minimal width with smallest surface area relaxed

points and 6 arcs is a set of width greater or equal than $\sqrt{2}/2$ (see Figure 5). Moreover, he observed that its volume and its surface are smaller than the ones of standard convex shapes of same minimal width (such the regular tetrahedron, the circular cone, the ball, Meissner's tetrahedron and Reuleaux's tetrahedron). Does this set minimise the volume and the surface among convex bodies of fixed minimal width ?

Due to the approximation made by (29), our algorithm does not always provide us a polytope which is precisely a member of $\mathcal{K}$. The resulting body satisfies only the width constraint at a discrete level. Notice that for a polytope,

$$\Delta = \min_{\nu \in \mathbf{S}^2} w_K(\nu),$$

is equal to the finite number of conditions

$$\min_{\nu_k} w_K(\nu_k) \tag{31}$$

deltapoly

where ν_k are the normal vectors of the polytope K. In order to get an element of $\mathcal{K}$, we apply the following post-processing: starting from the result of our optimisation process, we first compute its normal vectors. Then thanks to (23) and (31), the polytope defined by $\frac{1}{\Delta}P$ is in $\mathcal{K}$.

We present in figure 6 two different views of the resulting body. Hereafter are the values related to surfaces area and volumes of the our optimal shape and the body of E. Heil (for a minimal width equal to 1):

	Surface area	Volume
E. Heil body	2.9306	0.2983
Computed shape	2.9249	0.3862

table-newton

Notice that the polytope generated by our algorithm has a significantly smaller surface area than the shape proposed by E. Heil but has a greater volume.

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