QUALITATIVE AND NUMERICAL ANALYSIS OF A SPECTRAL PROBLEM WITH PERIMETER CONSTRAINT

1. Introduction

Many works adress the shape optimization problem

$$\min\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d, |\Omega| = 1\}, \tag{1.1}$$

where λ_k is the k-th eigenvalue of the Dirichlet Laplacian on Ω. Faber and Krahn proved that for $k = 1$ the minimizer is a ball of unit volume and Polya and Szego proved that for $k = 2$ the minimizer consists of two balls of volume one half. For the case $k \geq 3$ the shapes of the minimizers are unknown. Numerical studies of the optimal shapes were performed, initially, by E. Oudet [22] and, recently by P. Antunes, P. Freitas [2].

In recent articles [9],[12] authors switched from the measure constraint to a perimeter constraint:

$$\min\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d, \Omega \text{ open }, P(\Omega) = 1\}. \tag{1.2}$$

It is not difficult to see that problem (1.2) is equivalent to

$$\min\{\lambda_k(\Omega) + \operatorname{Per}(\Omega) : \Omega \subset \mathbb{R}^d, \Omega \text{ open}\} \tag{1.3}$$

in the sense that any solution of (1.2) is homothetic to a solution of (1.3) and conversely. We will use freely the two equivalent formulations.

In the case $k = 1$, the solution to problem (1.2) is obviously a ball as a consequence of the isoperimetric inequality and the Faber-Krahn inequality. The case $k = 2, d = 2$ was considered by D. Bucur, G. Buttazzo and A. Henrot in [9]. The authors provided existence, regularity, qualitative and numerical results. Recently G. De Plilippis and B. Velichkov [12] proved that the shape optimization problem (1.2) has a solution for any $k \in \mathbb{N}$ and for any dimension d. They also proved that the solution is bounded, connected, open with boundary which is $C^{1,\alpha}$ outside a closed set of Hausdorff dimension $d - 8$.

The numerical studies performed by É. Oudet [22] and P. Antunes, P. Freitas [2] for problem (1.1) show that the expected minimizers do not have an obvious geometric structure for $k \geq 5$. In [9] it is proved that the optimal shape Ω^* for $k = 2, d = 2$ does not contain any segment or any arc of circle in its boundary. This suggests that we cannot hope to find a simple geometric description of the solution of (1.2) even in the case of $k = 2$.

In this context it is relevant to introduce new numerical approaches which provide a precise description of optimal candidates in two and three dimensions. One numerical approach which has been used successfully in the last few years is the following parametric method. Considering the formulation (1.3) we note that the monotonicity of λ_k and the fact that in $\mathbb{R}^2$ convexification decreases perimeter imply that every solution of the problem (1.2) in the plane is convex. Thus we can represent an optimal candidate in the plane using its radial function $r(\theta)$. Furthermore, we can approximate the radial function r by its truncated Fourier series r_n (n sine and cosine coefficients). Doing this truncation, we don't perturb the eigenvalues too much. B. Osting gives an estimate of the error in [20]. In this way we can represent a good approximation of the boundary of a star convex shape by a finite number of parameters. It is possible to find the partial derivatives of $\lambda_k(\Omega_{r_n})$ with respect to the Fourier coefficients. Then a gradient descent algorithm can be used in order to find the optimal shape candidate in terms of first $2n+1$ Fourier coefficients. This method is very precise and gives reliable estimates of computed eigenvalues. The same method is used in [2]. The method also works in three dimensions and P. Antunes and P. Freitas announced a result in this direction. A possible drawback of using this method in three or more dimensions is the fact that we do not know apriori that the solutions of (1.2) are star-convex in dimension greater than two. Moreover, the implementation of this method in dimensions $d \geq 3$ is not straightforward.

A different approach consists of representing the shape Ω as a density function $\varphi : D \to [0, 1]$ (where D is a compact set of $\mathbb{R}^2$). In recent works of É. Oudet [23] and B. Bourdin, D. Bucur, É. Oudet [6] Γ-convergence results are used in order to approximate the perimeter of Ω and the eigenvalue $\lambda_k(\Omega)$ by relaxed functionals calculated on a density approximation of Ω.

The first main contribution of this article is proving that we can combine the two results above in order to produce a relaxation by Γ-convergence of $\lambda_k(\Omega) + \text{Per}(\Omega)$. This method works for $d = 2$ and $d = 3$ with results comparable to the first method. A similar method was used in the numerical study of an energy in connection with the Navier-Stokes equation plus a perimeter term in [15]. The advantage of this method is the fact that we do not make any topological assumption on the shapes we use. This method is not as precise as the first one due to the fact that we make a double approximation: of the shape Ω and of the cost functional.

As in other problems in the calculus of variations, finding an optimality condition, which must be satisfied by an optimizer, can help to understand better the solution, and derive some further properties. We would like to be able to write some optimality conditions for problem (1.2). The problem that arises is the fact that classical optimality conditions, using differential calculus, can only be written when the optimizer has a simple k-th eigenvalue. This is due to the fact that only simple eigenvalues are differentiable. The qualitative results obtained in [9] depend in a crucial way on the fact that the optimal shape in the case $k = 2, d = 2$ has a simple second eigenvalue. Our computations show that not all optimizers seem to have a simple eigenvalue, so the methods developed in [9] do not seem to apply in the general case. Multiplicity questions at the optimum are still open even for problem (1.1).

Numerically it was observed that as k increases, the numerical solution of (1.1) in the plane is such that λ_k is multiple and its multiplicity increases with k [2]. This phenomenon does not occur when we study problem (1.2), since we see that the optimizers for $k = 2, 6, 9, 13, 15$ probably have simple eigenvalues. We are let to believe that the qualitative results obtained in [9] should be true for values of k greater than 2. Since we do not have any theoretical information on the multiplicity at the optimum, it is not possible to use classical optimality conditions. Using similar methods to the ones developed by El Soufi and Ilias in [14] we are able to obtain some new optimality conditions for the minimizers of problem (1.2). These optimality conditions can be written regardless of the fact that the eigenvalue might be multiple at the optimum. As a consequence, we are able to obtain some qualitative information on solutioins of (1.2) for every k, d. In order to derive these optimality conditions, we assume that the optimizer of (1.2) is more regular. The actual known regularity is $C^{1,1}$ [12], but it is conjectured to be C^∞ like in [9].

2. Preliminaries

In the proof of our results we will need different theoretical tools, which are recalled below.

2.1. **Spectrum of a measurable set.** For well posedeness reasons, it is convenient to extend the notion of Sobolev space to any measurable set $\Omega \subset \mathbb{R}^N$ by defining

$$\widetilde{H}^1_0(\Omega) = \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ a.e. on } \Omega^c\}.$$

In general we have $H^1_0(\Omega) \subset \widetilde{H}^1_0(\Omega)$ and we have equality if, for instance, Ω has Lipschitz boundary. Furthermore, it is proved in [17, Chapter 4] that there exists a quasi-open set ω such that $\widetilde{H}^1_0(\Omega) = H^1_0(\omega)$.

For any $\Omega \subset \mathbb{R}^d$ of finite measure and any $f \in L^2$ we define $R_\Omega(f) \in \widetilde{H}^1_0(\Omega)$ as the weak solution in $\widetilde{H}^1_0(\Omega)$ of the equation

$$-\Delta u = f,\ u \in \widetilde{H}^1_0(\Omega)$$

or equivalently as the unique minimizer in $\widetilde{H}^1_0(\Omega)$ of

$$u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu.$$

Then R_Ω is a positive, self-adjoint and compact operator. As a consequence, its spectrum is discrete and its eigenvalues form a sequence converging to zero. Thus we can set

$$\lambda_k(\Omega) = \frac{1}{\Lambda_k(R_\Omega)}$$

where $0 \le ... \le \Lambda_k(\Omega) \le ... \le \Lambda_1(\Omega)$ are the eigenvalues of R_Ω.

If μ is a capacitary measure (i.e. $\mu(A) = 0$ if $\text{cap}(A) = 0$) then $\lambda_k(\mu)$ is defined as the k-th eigenvalue of the operator $-\Delta + \mu I$. The corresponding Rayleigh formulas are

$$\lambda_n(\mu) = \min_{E \in S_n} \max_{\phi \in E \setminus \{0\}} \frac{\int_D |\nabla \phi|^2 dx + \int_D \phi^2 d\mu}{\int_D \phi^2 dx},$$

where the minimum is taken over n dimensional subspaces of $H^1_0(D) \cap L^2(D;\mu)$. Using this formula we immediately deduce the following monotonicity property: if $\mu \le \nu$ then $\lambda_k(\mu) \le \lambda_k(\nu)$. We note that the eigenvalues of a shape Ω correspond to the eigenvalues of the measure $+\infty_{\Omega^c}$.

The notion of convergence which is well suited to the study of eigenvalue problems is γ-convergence. If $(\mu_n), \mu$ are capacitary measures we say that μ_n γ-converges to μ if

$$|R_{\mu_n} - R_\mu|_{\mathcal{L}(L^2(D))} \to 0.$$

We have denoted R_μ the resolvent of the operator $-\Delta + \mu I$. In particular, if μ_n γ-converges to μ, then

$$\lambda_k(\mu_n) \to \lambda_k(\mu).$$

A useful characterization of the γ-convergence of a sequence of sets (Ω_n) to another set Ω is the Mosco convergence of the spaces $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$. We suppose that Ω_n, Ω are contained in a bounded open set D. We say that $H^1_0(\Omega_n)$ converges to $H^1_0(\Omega)$ in the sense of Mosco if the two following conditions are satisfied:

- (M1) For all $\phi \in H^1_0(\Omega)$ there exists a sequence $\phi_n \in H^1_0(\Omega_n)$ such that ϕ_n converges strongly in $H^1_0(D)$ to ϕ.
- (M2) For every sequence $\phi_{n_k} \in H^1_0(\Omega_{n_k})$ weakly convergent in $H^1_0(D)$ to a function ϕ we have $\phi \in H^1_0(\Omega)$.

For more details we refer to [8, Chapter 6] and [17].

For every measurable set Ω of finite measure we denote w_Ω the weak solution of the equation

$$-\Delta w_\Omega = 1,\ w_\Omega \in \widetilde{H}^1_0(\Omega).$$

We have $w_U \le w_\Omega$ whenever $U \subset \Omega$ and

$$H^1_0(\{w_\Omega > 0\}) = \widetilde{H}^1_0(\{w_\Omega > 0\}) = \widetilde{H}^1_0(\Omega).$$

2.2. **Γ-convergence and Modica Mortola Theorem.** In shape optimization, many numerical methods replace the shape variable by some unknown function. One main difficulty in our context is to associate to this kind of functional framework a way to compute the perimeter of the set. To achieve this goal, the characteristic function χ_Ω will be approximated by a regular function $u \in H^1(\Omega)$ and the perimeter of Ω will be replaced by some smooth functional. The Γ-convergence result presented below, essentially due to Modica and Mortola [18], gives a satisfactory answer to this problem.

Definition 2.1. *Let X be a metric space and $F_\varepsilon, F : X \to [0,+\infty]$ a sequence of functionals on X (defined for $\varepsilon > 0$). We say that F_ε Γ-converges to F and we denote $F_\varepsilon \xrightarrow{\Gamma} F$ if the following two properties hold:*

(LI) *For every $x \in X$ and every $(x_\varepsilon) \subset X$ with $x_\varepsilon \to x$ we have*
$$F(x) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \tag{2.1}$$

(LS) *For every $x \in X$ there exists $(x_\varepsilon) \subset X$ such that $(x_\varepsilon) \to x$ and*
$$F(x) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon). \tag{2.2}$$

Given $x_0 \in X$ we will call *recovery sequence* a sequence (x_ε), which satisfies the (2.2) property. This sequence satisfies, in particular, the relation
$$\lim_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) = F(x).$$
Here are three main properties of the Γ-convergence.

Proposition 2.2. *If $F_\varepsilon \xrightarrow{\Gamma} F$ in X then the following properties hold:*

(i) *F is lower semicontinuous;*

(ii) *If $G : X \to [0,\infty)$ is a continuous functional then*
$$F_\varepsilon + G \xrightarrow{\Gamma} F + G.$$

(iii) *Suppose x_ε minimizes F_ε over X. Then every limit point of (x_ε) is a minimizer for F.*

The last property suggests that we could approximate a minimizer of F by a minimizer of F_ε for ε small enough. This method was successfully used in [6, 23].

Sometimes it is difficult to prove the (LS) property (2.2) for every $x \in X$. Having an element x with some good regularity properties may aid in constructing the recovery sequence. Therefore it is useful to find a dense set $\mathcal{D} \subset X \cap \{F < +\infty\}$ such that for every $x \in X \cap \{F < +\infty\}$ and $(u_n) \subset \mathcal{D}$, with $(u_n) \to x$ we have
$$\limsup_{n \to \infty} F(u_n) \leq F(x).$$
The result stated below is due to Modica and Mortola [18], and it provides an approximation of the perimeter using Γ-convergence.

Theorem 2.3. *Let D be a bounded open set and let $W : \mathbb{R} \to [0,\infty)$ be a continuous function such that $W(z) = 0$ if and only if $z \in \{0,1\}$. Denote $c = 2\int_0^1 \sqrt{W(s)}ds$. We define $F_\varepsilon, F : L^1(D) \to [0,+\infty]$ by*
$$F_\varepsilon(u) = \begin{cases} \varepsilon \int_D |\nabla u|^2 + \frac{1}{\varepsilon}\int_D W(u) & u \in H^1(D) \\ +\infty & \textit{otherwise} \end{cases}$$
and
$$F(u) = \begin{cases} c\,\mathrm{Per}(u^{-1}(1)) & u \in BV(D;\{0,1\}) \\ +\infty & \textit{otherwise} \end{cases}$$
then
$$F_\varepsilon \xrightarrow{\Gamma} F$$
in the $L^1(D)$ topology.

For a proof we refer to [1] or [10]. In the numerical simulations we fix the potential
$$W(s) = s^2(1-s)^2$$
which imposes the corresponding constant is $c = 1/3$.

Remark 2.4. In general if $F_\varepsilon \xrightarrow{\Gamma} F$ and $G_\varepsilon \xrightarrow{\Gamma} G$ we cannot conclude that

$$F_\varepsilon + G_\varepsilon \xrightarrow{\Gamma} F + G.$$

Thus, the result proved in Section 3 is not trivial. One sufficient condition for the above implication to hold would be that for each u we could find the same recovery sequence for F and G. For more details and examples see [7].

Later on we will use the recovery sequence obtained in the proof of Theorem 2.3, i.e the sequence which satisfies the (2.2) property in the approximation of the perimeter. This sequence will help us to construct the recovery sequence in our Γ-convergence result. In the case where φ is a characteristic function of a set with finite perimeter, i.e $\varphi = \chi_\Omega \in BV(D)$, having smooth boundary and $\mathcal{H}^{n-1}(\partial\Omega \cap \partial D) = 0$, we define:

$$\psi_\varepsilon(t) = \int_0^t \frac{\varepsilon}{\sqrt{\varepsilon + W(s)}} ds$$

$$\eta_\varepsilon(t) = \begin{cases} 1 & t \leq 0 \\ 1 - \psi_\varepsilon^{-1}(t) & 0 \leq t \leq \psi_\varepsilon(1) \\ 0 & t \geq \psi_\varepsilon(1) \end{cases}$$

$$\varphi_\varepsilon(x) = \eta_\varepsilon(d_\Omega(x)), \text{ for } x \in D. \tag{2.3}$$

where d_Ω is the usual signed distance function to $\partial\Omega$. Note that if $d(x, \partial\Omega) \notin [0, \sqrt{\varepsilon}]$ then $\varphi_\varepsilon(x) = \chi_\Omega$.

2.3. **Perturbation theory for eigenvalues.** Let (f_ε) be a family of diffeomorphisms of $\mathbb{R}^d$ which depend analytically of ε, such that f_0 is the identity. Each such family of diffeomorphisms determines a sequence of perturbations $(\Omega_\varepsilon) = (f_\varepsilon(\Omega))$ of Ω. The vector field $V = \frac{d}{d\varepsilon} f_\varepsilon|_{\varepsilon=0}$ is called the direction of the perturbation. One natural question is to see whether the map

$$\varepsilon \mapsto \lambda_k(\Omega_\varepsilon) \tag{2.4}$$

is differentiable at $\varepsilon = 0$. It is known that the above map is differentiable if and only if $\lambda_k(\Omega)$ is simple. Nevertheless, it is possible to prove that if $\lambda_k(\Omega)$ has multiplicity $p > 1$ and if we consider an analytic perturbation $\Omega_\varepsilon = f_\varepsilon(\Omega)$, then the p corresponding eigenvalues move on p smooth curves as ε varies. The differentiability is lost because the p eigenvalues change their places on the p smooth curves as ε passes through zero, due to their ordering. We could recover some informations on differentiability if we relabel them. This method has been used in [14]. We present below some of the results needed to derive our optimality conditions.

Consider Ω a bounded, open set of class C^3 in $\mathbb{R}^N$; therefore the mean curvature $\mathcal{H}$ is well defined and continuous. We denote by n the outer normal to Ω. Any perimeter perturbation $\Omega_\varepsilon = f_\varepsilon(\Omega)$ induces a function $v = \langle \frac{d}{d\varepsilon} f_\varepsilon|_{\varepsilon=0}, n\rangle$ on $\partial\Omega$ satisfying $\int_{\partial\Omega} \mathcal{H}\, v\, d\sigma = 0$. We denote by $\mathcal{P}_0(\partial\Omega)$ the set of C^1 functions on $\partial\Omega$ such that $\int_{\partial\Omega} \mathcal{H}\, v\, d\sigma = 0$. We denote by $\operatorname{div}_\Gamma$ the tangential divergence with respect to Γ. We refer to [17, Section 5.4.3], for a precise description of $\operatorname{div}_\Gamma$.

Lemma 2.5. *Let $v \in \mathcal{P}_0(\partial\Omega)$. Then there exists an analytic perimeter preserving deformation $\Omega_\varepsilon = f_\varepsilon(\Omega)$ such that $v = \langle \frac{d}{d\varepsilon} f_\varepsilon|_{\varepsilon=0}, n\rangle$.*

Proof: Let U be an open neighborhood of $\overline{\Omega}$ and $\tilde{v}, \tilde{n}$ be smooth extensions of v, n to U. For ε sufficiently small, the map $\varphi_\varepsilon(x) = x + \varepsilon\tilde{v}(x)\tilde{n}(x)$ is a diffeomorphism from Ω to $\varphi_\varepsilon(\Omega)$ (local inversion theorem). This deformation is analytic in ε, but is not necessarily perimeter-preserving.

Let X be an analytic vector field on U such that $\int_{\partial\Omega} \operatorname{div}_{\partial\Omega} X \neq 0$ and let ϕ_t be the one parameter group of diffeomorphisms associated to X. Define $(t, \varepsilon) \mapsto G(t, \varepsilon) = \operatorname{Per}(\phi_t \circ$

$\varphi_\varepsilon(\Omega))$. Using the fact that $\frac{d\phi_t}{dt}|_{t=0} = X$ and Proposition 5.4.18 from [17] we obtain

$$\frac{\partial G}{\partial t}(0,0) = \frac{d}{dt}\operatorname{Per}(\phi_t(\Omega)) = \int_{\partial\Omega} \operatorname{div}_{\partial\Omega} X d\sigma \neq 0.$$

Therefore we can apply the implicit function theorem around $(0,0)$ to see that there exists an analytic function $\varepsilon \mapsto t(\varepsilon)$ defined on a neighborhood $(-\eta,\eta)$ of 0 such that

$$G(t(\varepsilon),\varepsilon) = G(0,0) = \operatorname{Per}(\Omega).$$

Thus the deformation $g_\varepsilon = \phi_{t(\varepsilon)} \circ \varphi_\varepsilon$ is perimeter preserving. Moreover, using Propositions 5.4.9 and 5.4.18 from [17], we have

$$t'(0) = -\frac{\frac{d}{d\varepsilon}\operatorname{Per}(\varphi_\varepsilon(\Omega))|_{\varepsilon=0}}{\frac{d}{dt}\operatorname{Per}(\phi_t(\Omega))|_{t=0}} = -\frac{\int_{\partial\Omega}\operatorname{div}_{\partial\Omega}\tilde{v}\tilde{n}d\sigma}{\int_{\partial\Omega}\operatorname{div}_{\partial\Omega} X d\sigma} = -\frac{\int_{\partial\Omega}\mathcal{H}\ v\ d\sigma}{\int_{\partial\Omega}\operatorname{div}_{\partial\Omega} X d\sigma} = 0.$$

Therefore, if we set $H(t,\varepsilon) = \phi_t \circ \varphi_\varepsilon$ then

$$\frac{d}{d\varepsilon}g_\varepsilon(x)|_{\varepsilon=0} = \frac{d}{dt}H(t(0),0)t'(0) + \frac{d}{d\varepsilon}H(t(0),0) = \frac{d\varphi_\varepsilon}{d\varepsilon}|_{\varepsilon=0} = \tilde{v}(x)\tilde{n}(x) = v(x)n(x)$$

for $x \in \partial\Omega$. In conclusion, g_ε is the desired perturbation. □

Below we present two results from [14], which will be used freely in the rest of the article. We omit the proofs, as they can be found in the cited article.

Lemma 2.6. *Let λ be an eigenvalue of multiplicity p of the Dirichlet Laplacian on Ω. For any analytic deformation Ω_ε of Ω there exist p families of real numbers $(\Lambda_{i,\varepsilon})_{i\le p}$ and p families of functions $(\phi_{i,\varepsilon})_{i\le p} \subset C^\infty(\Omega_\varepsilon)$, depending analytically on ε, satisfying for all $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ and for all $i \in \{1,...,p\}$:*

(a) $\lambda_{i,0} = \lambda$.
(b) *The family $\{\phi_{1,\varepsilon},...,\phi_{p,\varepsilon}\}$ is orthonormal in $L^2(\Omega_\varepsilon)$.*
(c) *We have* $\begin{cases}\Delta\phi_{i,\varepsilon} = \Lambda_{i,\varepsilon}\phi_{i,\varepsilon} \text{ in } \Omega_\varepsilon\\ \phi_{i,\varepsilon} = 0 \text{ on } \partial\Omega_\varepsilon.\end{cases}$

Lemma 2.7. *Let λ be an eigenvalue of multiplicity p of the Dirichlet Laplace operator and denote E_λ the corresponding eigenspace. Let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be an analytic deformation of Ω. Let $(\Lambda_{i,\varepsilon})_{i\le p}$ and $(\phi_{i,\varepsilon})_{i\le p}$ be like in Lemma 2.6. Then $\Lambda_i' = \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}|_{\varepsilon=0}$ are the eigenvalues of the quadratic form q_v defined on $E_\lambda \subset L^2(\Omega)$ by*

$$q_v(\phi) = -\int_{\partial\Omega}\left(\frac{\partial\phi}{\partial n}\right)^2 v\ d\sigma,$$

where $v = \langle\frac{d}{d\varepsilon}f_\varepsilon, n\rangle$. Moreover, the L^2-orthonormal basis $\phi_{1,0},...,\phi_{p,0}$ diagonalizes q_v on E_λ.

We define the following notion of critical domain for the eigenvalues of the Dirichlet Laplacian, which generalizes the notion of local minimum or local maximum.

Definition 2.8. *The domain Ω is said to be* critical *for the k-the eigenvalue of the Dirichlet problem if, for any analytic perimeter-preserving deformation Ω_ε of Ω, the right-sided and left-sided derivatives of λ_k,ε (see Lemma 2.6) at $\varepsilon = 0$ have opposite signs, that is*

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} \times \frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} \le 0.$$

3. Γ-convergence result

In this section we construct a Γ-convergence approximation for $\lambda_k(\Omega) + \mathrm{Per}(\Omega)$. This result allows us to construct a numerical method for the study of problem (1.2), which will be presented in the next section. Consider $F : \mathbb{R}^k \to \mathbb{R}_+$ a continuous function which is increasing in each variable. Let $D \subset \mathbb{R}^N$ be a bounded, open set. For every $\varphi : D \to \mathbb{R}_+$, measurable we define $\lambda_k(\varphi) = \lambda_k(\varphi\ dx)$, where $\varphi\ dx$ is seen as a capacitary measure. In the following, q will be a fixed positive real number.

Theorem. Define $J_\varepsilon : L^1(D) \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$J_\varepsilon(\varphi) = F\left(\lambda_1\left(\frac{1-\varphi}{\varepsilon^q}dx\right), ..., \lambda_k\left(\frac{1-\varphi}{\varepsilon^q}dx\right)\right) + \varepsilon\int_D |\nabla\varphi|^2 + \frac{1}{\varepsilon}\int_D \varphi^2(1-\varphi)^2$$

if $\varphi \in H^1(D)$ and $+\infty$ otherwise. Then $J_\varepsilon \xrightarrow{\Gamma} J$ in the $L^1(D)$ topology, where

$$J(\varphi) = \begin{cases} F(\lambda_1(\Omega), ..., \lambda_k(\Omega)) + \frac{1}{3}\mathrm{Per}(\Omega), & \text{if } \varphi = \chi_\Omega \in BV(D) \\ +\infty & \text{otherwise} \end{cases}$$

Proof: For simplicity, in the rest of the proof we denote the quantity $F(\lambda_1(\Omega), .., \lambda_k(\Omega))$ by $F(\Omega)$. We make the same convention when instead of Ω we have a measure μ. Let us begin by proving the $\Gamma - \limsup$ part of our result.

1. Reduction to regular domains. In order to construct a recovery sequence we perform the operation described in the Preliminaries section. It suffices to work on a dense set $\mathcal{D}$ of $\{F < +\infty\}$ and to prove that for each $\Omega \in \{F < +\infty\}$ we can find $\Omega_n \in \mathcal{D}$ such that $\chi_{\Omega_n} \to \chi_\Omega$ in L^1 topology and $\limsup_n F(\chi_{\Omega_n}) \leq F(\chi_\Omega)$.

In [3], Thm 3.4.2 it is proved that we can choose $\mathcal{D}$ to be the family of subsets of D with finite perimeter and smooth boundary. If φ is the characteristic function χ_Ω of Ω and it belongs to $BV(D)$ then Ω is a set of finite perimeter. The theorem we cited above says that each finite perimeter Ω set can be approximated in the $L^1(D)$ topology with a sequence (Ω_n) of finite perimeter sets having smooth boundary such that $\mathrm{Per}(\Omega_n) \to \mathrm{Per}(\Omega)$. At this point it is not clear if we have $\limsup\limits_{n\to\infty} F(\Omega_n) \leq F(\Omega)$. The objective of the following paragraphs is to construct (Ω_n) in such a way that the above inequality holds.

If we denote (ρ_k) a sequence of mollifiers, we have

$$\begin{aligned} \mathrm{Per}(\Omega) &= \int_{\mathbb{R}^N} |D\chi_\Omega| = \lim_{k\to\infty}\int_{\mathbb{R}^N} |\nabla\chi_\Omega * \rho_k| = \\ &= \lim_{k\to\infty}\int_0^1 \mathrm{Per}(\{\chi_\Omega * \rho_k > t\})dt \geq \int_0^1 \liminf_{k\to\infty} \mathrm{Per}(\{\chi_\Omega * \rho_k > t\})dt \end{aligned} \tag{3.1}$$

where we have applied the co-area formula and Fatou's lemma. By applying Chebyshev's inequality we obtain that

$$|\{\chi_\Omega * \rho_k > t\} \setminus \Omega| = |\{\chi_\Omega * \rho_k - \chi_\Omega \geq t\}| \leq \frac{1}{t}\int_{\mathbb{R}^N} |\chi_\Omega * \rho_k - \chi_\Omega|$$

and

$$|\Omega \setminus \{\chi_\Omega * \rho_k > t\}| = |\{\chi_\Omega - \chi_\Omega * \rho_k \geq 1-t\}| \leq \frac{1}{1-t}\int_{\mathbb{R}^N} |\chi_\Omega * \rho_k - \chi_\Omega|.$$

Therefore $\chi_{\{\chi_\Omega * \rho_k > t\}}$ converges to χ_Ω in the $L^1(D)$ topology for almost every $t \in (0,1)$. By the lower semicontinuity of the perimeter we deduce that

$$\liminf_{k\to\infty} \mathrm{Per}(\{\chi_\Omega * \rho_k > t\}) \geq \mathrm{Per}(\Omega).$$

Combining this with (3.1) we obtain

$$\liminf_{k\to\infty} \mathrm{Per}(\{\chi_\Omega * \rho_k > t\}) = \mathrm{Per}(\Omega).$$

for almost every $t \in (0,1)$. Sard's theorem tells us that the level sets of $\chi_\Omega * \rho_k$ are smooth for almost every t. Moreover, Lemma 2.95 from [3] tells us that almost all level sets of $\chi_M * \rho_k$ are transversal, i.e. $\mathcal{H}^{n-1}(\partial\{\chi_M * \rho_k\} \cap \partial D) = 0$. In this way, we can choose the smooth, transversal approximating sets at almost every level $t \in (0,1)$.

Denote $w = R_\Omega(1) = R_\omega(1)$ where $\omega \subset \Omega$ is a quasi open set with the property that $H_0^1(\omega) = \widetilde{H}_0^1(\Omega)$. We can assume that $\|w\|_\infty \le 1$ (or otherwise rescale it) so that we get $w \le \chi_\Omega$ which implies that $w * \rho_k \le \chi_\Omega * \rho_k$ and as a consequence $\{w*\rho_k > t\} \subseteq \{\chi_\Omega * \rho_k > t\}$.

We want to prove that $\limsup_{k\to\infty} F(\{w * \rho_k > t\}) \le F(\{w > t\})$. Denote $A_k = \{w * \rho_k > t\} \cap \{w > t\}$. It is enough to prove that (A_k) γ-converges to $\{w > t\}$. As a matter of fact, if this holds, then

$$\limsup_{k\to\infty} F(\{w * \rho_k > t\}) \le \lim_{k\to\infty} F(A_k) = F(\{w > t\})$$

To prove this γ-convergence result it suffices to prove the first Mosco condition, since the second one comes from $A_k \subset \{w > t\}$. For more details we refer to [8, Section 4.5]. To prove the first Mosco condition it is enough to prove it on a dense subset of $H_0^1(\{w > t\})$. One such dense subset is given in [11] Prop 5.5 and is $\{C_c^\infty(\mathbb{R}^N) \cdot (w-t)^+\}$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then if $\varphi_k = \varphi \cdot \min\{(w * \rho_k - t)^+, (w-t)^+\}$ we have $\varphi_k \to \varphi \cdot (w-t)^+$ in $H_0^1(D)$ and $\varphi_k \in H_0^1(A_k)$. This concludes the proof of the fact that A_k γ-converges to $\{w > t\}$.

Therefore we have found a sequence

$$B_k^t = \{w * \rho_k > t\} \subseteq C_k^t = \{\chi_M * \rho_k > t\}$$

with $C_k^t \to \chi_\Omega$ in $L^1(D)$, $\liminf_{k\to\infty} \operatorname{Per}(C_k^t) = \operatorname{Per}(\Omega)$ for almost every t, $F(C_k^t) \le F(B_k^t)$ and

$$\limsup_{k\to\infty} F(B_k^t) \le F(\{w > t\}).$$

Thus, we can choose a diagonal sequence $E_k = C_k^{t_k}$ with $t_k \to 0$ such that $\chi_{E_k} \to \chi_\Omega$ in $L^1(D)$, $\operatorname{Per}(E_k) \to \operatorname{Per}(\Omega)$ in order to obtain

$$\limsup_{k\to\infty} F(E_k) \le F(\{w > 0\}) = F(\omega) = F(\Omega).$$

2. Proof of the $\Gamma - \limsup$ part. Using the previous density result, it suffices to prove the $\Gamma - \limsup$ only for characteristic functions of smooth sets with finite perimeter. Let $\varphi \in L^1(D)$ with $J(\varphi) < +\infty$. Then φ is the characteristic function of a set Ω with finite perimeter. We assume, as mentioned above, that Ω has smooth boundary.

We recall that the recovery sequence (2.3) in the Γ-limit approximation of the perimeter for a smooth set Ω with $\mathcal{H}^{n-1}(\partial\Omega \cap \partial D) = 0$ can be choosen such that $\chi_\Omega(x) = \varphi_\varepsilon(x)$ for $d_\Omega(x) \notin [0, \sqrt{\varepsilon}]$.

We take $\varphi_\varepsilon \in H^1(D)$ the recovery sequence (2.3). Then we have $\varphi_\varepsilon \to \varphi$ in $L^1(D)$ and

$$\lim_{\varepsilon\to 0} \left[\varepsilon \int_D |\nabla \varphi_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_D \varphi_\varepsilon^2 (1-\varphi_\varepsilon^2) dx\right] = \frac{1}{3} \operatorname{Per}(\Omega).$$

Since for $x \in \Omega$ we have $\varphi_\varepsilon(x) = 1$ we have $+\infty_{D\setminus\Omega} \ge \dfrac{1-\varphi_\varepsilon}{\varepsilon^q}$ and by the monotonicity of λ_j we have

$$\lambda_j(\Omega) = \lambda_j(+\infty_{D\setminus\Omega}) \ge \lambda_j\left(\frac{1-\varphi_\varepsilon}{\varepsilon^q} dx\right).$$

Using the monotonicity of F we obtain that

$$\limsup_{\varepsilon\to 0} F\left(\frac{1-\varphi_\varepsilon}{\varepsilon^q} dx\right) \le F(\Omega).$$

3. Proof of the $\Gamma - \liminf$ part. Let $\varphi \in L^1(D)$ and $(\varphi_\varepsilon) \in L^1(D)$ such that $\varphi_\varepsilon \to \varphi$ in $L^1(D)$. We assume that $\liminf_{\varepsilon\to 0} J_\varepsilon(\varphi) < +\infty$ since otherwise the result is obvious. The

$\Gamma - \liminf$ part of the Modica-Mortola theorem tells us that

$$+\infty > \liminf_{\varepsilon\to 0} \varepsilon \int_D |\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\int_D \varphi_\varepsilon^2(1-\varphi_\varepsilon)^2 \geq \frac{1}{3}\int_D |D\varphi| = \frac{1}{3}\operatorname{Per}(\Omega)$$

which implies that $\varphi \in BV(\Omega)$ and that Ω has finite perimeter in D. It suffices to prove that

$$\liminf_{\varepsilon\to 0} F\left(\frac{1-\varphi_\varepsilon}{\varepsilon^q}dx\right) \geq F(\Omega),$$

which reduces to proving that

$$\liminf_{\varepsilon\to 0} \lambda_i\left(\frac{1-\varphi_\varepsilon}{\varepsilon^q}dx\right) \geq \lambda_i(\Omega).$$

Let w_ε be the solution of

$$\begin{cases} -\Delta w_\varepsilon + \frac{1-\varphi_\varepsilon}{\varepsilon^q} w_\varepsilon = 1 & \text{in } D \\ w_\varepsilon \in H_0^1(D). \end{cases}$$

Without loss of generality we can replace $\liminf$ with $\lim$ by taking a sequence ε_k which realizes the $\liminf$. Denoting $\varphi_k = \varphi_{\varepsilon_k}$, we have to prove that

$$\lim_{n\to\infty} \lambda_i\left(\frac{1-\varphi_k}{\varepsilon_k^q}dx\right) \geq \lambda_i(\Omega).$$

By compactness there is a subsequence of (w_{n_k}) converging weakly in $H_0^1(D)$ to w. We can choose a subsequence of this sequence which converges almost everywhere to w. For simplicity we relabel this subsequence (w_k). It is enough to prove the inequality for (φ_k) (the corresponding functions for this new sequence (w_k)).

Taking w_k as test functions in the partial differential equation we get

$$\int_D \frac{1-\varphi_k}{\varepsilon_k^q} w_k^2 = \int_D w_k - \int_D |\nabla w_k|^2 \leq \int_D w_k \leq \int_D w_D.$$

We know that

$$\liminf_{k\to\infty} \frac{1-\varphi_k(x)}{\varepsilon_k^q} = +\infty$$

for $x \in \Omega^c$ since $1-\varphi_k(x) \to 1$ a.e. on Ω_c and $\varepsilon_k \to 0^+$. Therefore since $w_k \to w$ almost everywhere, if $w(x) > 0$, $x \notin \Omega$ and $w_k(x) \to w(x)$ then

$$\liminf_{k\to\infty} \frac{1-\varphi_k(x)}{\varepsilon_k^q} w_k^2(x) = +\infty.$$

Fatou's Lemma tells us that

$$+\infty > \liminf_{k\to\infty} \int_D \frac{1-\varphi_k}{\varepsilon_k^q} w_k^2 \geq \int_D \liminf_{k\to\infty} \frac{1-\varphi_k}{\varepsilon_k^q} w_k^2 \geq \int_{\Omega^c} \liminf_{k\to\infty} \frac{1-\varphi_k}{\varepsilon_k^q} w_k^2$$

This inequality and the previous remarks impliy that the set $\Omega^c \cap \{w > 0\}$ is of measure zero, and therefore $w \in \widetilde{H}_0^1(\Omega)$. Since the γ-convergence is compact, up to a subsequence we have

$$\mu_\varepsilon = \frac{1-\varphi_k}{\varepsilon_k^q} \xrightarrow{\gamma} \mu \geq +\infty_{\Omega^c}.$$

As a consequence, we have

$$\lim_{k\to\infty} \lambda_i\left(\frac{1-\varphi_k}{\varepsilon_k^q}dx\right) \geq \lambda_i(\Omega),$$

which finishes the proof of the $\Gamma - \liminf$ part. $\square$

4. Numerical study of problem (1.2)

The method we developed for studying problem (1.2) combines the Γ-convergence methods used in approximating the perimeter (used in [23]) and the eigenvalues of the Laplace operator (used in [6]). The combination of the two cited methods is made possible by the Γ-convergence result proved in the previous section. As it has been underlined, our Γ-convergence method is very flexible with respect to both the dimension and the topology of the shapes. In order to evaluate the quality of our solution we recall in subsection 4.2 the method used successfully by B. Osting [20] and P. Antunes, P. Freitas [2]. In Table 1 we illustrate that both methods give the same results in the easy context of the two dimensional case. Finally, we extend previous results in the three dimensional case, where we some of the optimal shapes found seem to be non-convex. This behaviour has been conjectured in [9].

4.1. **Method based on the Γ-convergence result.** We relax our shape optimization problem with respect to Ω by an optimization problem of an unknown function $\varphi : D \to [0,1]$. In our computations we choose $D=[0,a]^2$ and imposed periodic conditions (so that the perimeter of Ω would not be influenced by the boundary of D). We consider a $N\times N$ uniform grid and we represent the function φ by its values $(\varphi_{i,j})_{i,j=1}^N$ on this grid. We approximate

$$\varphi \mapsto \varepsilon \int_D |\nabla \varphi|^2 + \frac{1}{\varepsilon}\int_D \varphi^2(1-\varphi)^2$$

by using centred finite differences on the considered grid. This approximation is equivalent to considering a piecewise linear function associated to the grid values.

For the eigenvalue approximation we have to discretize the problem:

$$-\Delta u_k + \frac{1-\varphi}{\varepsilon^2} u_k = \lambda_k u_k.$$

To obtain a matrix formulation, we fix an ordering on the $N\times N$ grid. We denote by $\bar\psi$ the vector which contains the values on the grid of the function ψ with respect to this fixed ordering. We define A to be the $N^2\times N^2$ matrix associated to the discrete Laplacian on the considered grid, with respect to the fixed ordering. The discretized eigenvalue problem becomes

$$\left[A + \frac{1-\bar\varphi}{\varepsilon^2} I\right] \bar u_k = \lambda_k \bar u_k.$$

We used the Matlab solver `eigs` to solve this matrix eigenvalue problem. The expression of the discrete gradient of our functional with respect to each component of $\bar\varphi$ is

$$-\frac{1}{\varepsilon^2}\bar u_k^2.$$

We refer to [6] for more details.

We can compute the gradient of $\varphi \mapsto \varepsilon \int_D |\nabla\varphi|^2 + \frac{1}{\varepsilon}\int_D \varphi^2(1-\varphi)^2$ with respect to a perturbation θ of $\varphi \in H^1(D)$ as follows:

$$\begin{aligned}
&\frac{d}{dt}\left[\varepsilon \int_D |\nabla(\varphi+t\theta)|^2 + \frac{1}{\varepsilon}\int_D (\varphi+t\theta)^2(1-(\varphi+t\theta))^2\right]_{t=0} = \\
&= 2\varepsilon \int_D \langle \nabla\varphi, \nabla\theta\rangle + \frac{1}{\varepsilon}\int_D (2\varphi - 6\varphi^2 + 4\varphi^3)\theta \\
&= \int_D \left[-2\varepsilon\Delta\varphi + \frac{1}{\varepsilon}(2\varphi - 6\varphi^2+4\varphi^3)\right]\theta
\end{aligned}$$

Thus the discrete gradient of $\varphi \mapsto \varepsilon \int_D |\nabla\varphi|^2 + \frac{1}{\varepsilon}\int_D \varphi^2(1-\varphi)^2$ with respect to φ is given by

$$-2\varepsilon(4\bar\varphi_{i,j} - \bar\varphi_{i+i,j} - \bar\varphi_{i-1,j} - \bar\varphi_{i,j+1} - \bar\varphi_{i,j-1}) + \frac{1}{\varepsilon}(2\bar\varphi_{i,j} - 6\bar\varphi_{i,j}^2 + 4\bar\varphi_{i,j}^3). \tag{4.1}$$

To obtain a solution φ_0 of the problem

$$\min \left[\varepsilon_0 \int_D |\nabla \varphi|^2 + \frac{1}{\varepsilon_0} \int_D \varphi^2(1-\varphi)^2 + \lambda_k(\frac{1-\varphi}{\varepsilon_0^2} dx) \right]$$

we start from a random configuration with a concentration around the center of the grid. Numerical experiments have shown that starting from a totally random configuration tends to lead to a shape consisting of k disks. This configuration is a local minima, but not the global one, since we know that the optimal shape is connected [12]. We think this behaviour is due to the fact that when we approximate Ω by density functions, the optimization of λ_k tends to separate Ω into nodal domains. Then the perimeter, which is optimized locally, transforms those domains into disks. This observation motivates our previous initialization. For the optimization part, we used the quasi-Newton algorithm LBFGS implemented in [24],[25].

The choice of the initial parameter ε_0 is important for the algorithm to converge. As in phase field methods, it is well known that if we choose ε_0 too small, then the term $\frac{1}{\varepsilon} \int_D u^2(1-u)^2 dx$ will be too strong, and forces u to immediately become a characteristic function. If ε_0 is too large, then the shape will be too diffuse. Numerical experiments have shown that $\varepsilon_0 \in [\frac{1}{N}, \frac{4}{N}]$ are suitable for obtaining the expected results. This observation is well known in the phase-field community.

Algorithm 1 General form of optimization algorithm for $\min_\varphi J_\varepsilon(\varphi)$

Require: $k \in \mathbb{N}, \varepsilon_0 > 0, p_{max} \in \mathbb{N}, N \in \mathbb{N}, \omega \in (0,1), \mathrm{tol} \in (0,1)$

$\varepsilon = \varepsilon_0$;
Choose a random initial shape φ concentrated around the center of D;
repeat
 $p = 1$;
 repeat
  Compute the eigenpair (λ_k, u_k) of $A + \frac{1-\bar{\varphi}}{\varepsilon^2} I$ and the gradient $\nabla \lambda_k(\varphi) = -\frac{1}{\varepsilon^2} \bar{u}_k$;
  Compute the gradient of $\varphi \mapsto \varepsilon \int_D |\nabla \varphi|^2 + \frac{1}{\varepsilon} \int_D \varphi^2 (1-\varphi)^2$ with respect to the components of $\bar{\varphi}$ on the grid using formula (4.1);
  Do a step of the LBFGS algorithm: update descent direction and do a linesearch;
  $\varphi \leftarrow \varphi - d_p$;
  $p \leftarrow p + 1$;
 until $p = p_{max}$ or $|d_p| < \mathrm{tol}$;
 $\varepsilon = (1-\omega)\varepsilon$;
until $\varepsilon < 1/N$.

4.2. **The approach of B. Osting** [20] **and P. Antunes, P. Freitas** [2]**.** In order to verify our results, we compare them with the ones obtained using the boundary parametrization method mentioned in the introduction. This method is well known, and was applied in [2],[20] and [21]. We present it below for the sake of completeness.

We know that the solutions to problem (1.2) in $\mathbb{R}^2$ are convex shapes, so every such shape is uniquely defined by its radial function $r(\theta)$, $\theta \in [0, 2\pi)$. B. Osting proved in [20, Prop. 3.1] that the error $|\lambda_k(\Omega_r) - \lambda_k(\Omega_{r_n})|$ can be made arbitrarily small if we choose n big enough, where r_n is the truncation of the Fourier series representation of r to $2n+1$ coefficients:

$$r_n(\theta) = a_0 + \sum_{k=1}^n a_k \cos(k\theta) + \sum_{k=1}^n b_k \sin(k\theta).$$

This allows us to write $\lambda_k(\Omega)$ as a function of $2n+1$ variables $\lambda_k(a_0, a_1, ..., a_n, b_1, ..., b_n)$. Furthermore, using the fact that the derivative of $\lambda_k(\Omega)$ with respect to a perturbation V

of the boundary is

$$\frac{d\lambda_k(\Omega)}{dV} = -\int_{\partial\Omega} \left(\frac{\partial u_k}{\partial n}\right)^2 (V.n) d\sigma$$

(proofs and other references can be found in [16, 17]) we can find that

$$\frac{\partial \lambda_k}{\partial a_k} = -\int_0^{2\pi} r(\theta)\cos(k\theta) \left| \frac{\partial u}{\partial n}(\rho(\theta),\theta) \right|^2 d\theta$$

$$\frac{\partial \lambda_k}{\partial b_k} = -\int_0^{2\pi} r(\theta)\sin(k\theta) \left| \frac{\partial u}{\partial n}(\rho(\theta),\theta) \right|^2 d\theta.$$

We can find similar formulas for the derivatives of the perimeter in terms of Fourier coefficients. For computing the eigenvalues and normal derivatives of the eigenfunctions it is possible to use the publicly available software MpsPack [4].

4.3. **Our numerical results.** In order to solve numerically problem (1.2), in its equivalent form (1.3), we search the solutions of the relaxed problem

$$\min \left[\varepsilon_0 \int_D |\nabla \varphi|^2 + \frac{1}{\varepsilon_0} \int_D \varphi^2 (1-\varphi)^2 + \lambda_k (\frac{1-\varphi}{\varepsilon_0^2} dx) \right]$$

We use the method presented in subsection 4.1 on the square $D = [0,a]^2$ (where a is chosen such that the solution of (1.3) fits inside D).

Since the method presented in subsection 4.2 was used successfully in the study of the problem (1.1), we employ it to find the numerical solutions of (1.2). These solutions consist a benchmark to which we compare the results we found using our Γ-convergence methods.

The optimal shapes obtained with the Γ-convergence method coincide with the ones found using the boundary parametrization method. The numerical results can be seen in Figure 1. To compare the accuracy of the results, we took the optimal shapes obtained with the Γ-convergence method and we isolated the 0.5 level set. We choose a point in its convex hull, the centroid G of a discretization $\{x_1, ..., x_l\}$ of the boundary, and computed the distances from that point to the contour, denoted by $\{\rho_1, ..., \rho_l\}$ as well as the angles made by Gx_i with the positive x-axis, denoted by $\{\theta_1, ..., \theta_l\}$. This procedure gives us a radial parametrization of our domain and using a least squares fit

$$\min_{(a_j)_{j=0}^n, (b_j)_{j=0}^n} \sum_{i=1}^l \left(a_0 + \sum_{j=1}^n a_j \cos(j\theta_i) + \sum_{j=1}^n \sin(j\theta_i) - \rho_i \right)^2$$

we are able to find the first $2n+1$ Fourier coefficients of this radial function. We use these coefficients to construct the radial function of our shape Ω^*. We use MpsPack to compute $\lambda_k(\Omega^*) + \mathrm{Per}(\Omega^*)$ and we compare the results, which can be seen in Table 1. We can see that the results agree, and in general the ones obtained with the Γ-convergence method are a bit weaker, in the sense that the minimal value is higher. Still, the fact that we obtain the same shapes, with small errors, shows that the Γ-convergence method is a suitable tool for the study of problem (1.2). Furthermore, it gets close enough to the optimizer without imposing any topological constraints.

One interesting question that has been addressed in several papers ([2],[22]) is the multiplicity of λ_k at the optimum. We noticed in our computations that the optimal shape for (1.3) does not always have multiple k-th eigenvalue. This was already proved for $k = 2$ in [9] and our computations have shown that for $k = 6, 9, 13, 15$ the optimal eigenvalues should be simple. This behaviour is different from the one observed for problem (1.3). It is known that if a local minimizer of problem (1.1) would have simple eigenvalue then its eigenfunction would satisfy the overdetermined problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = c & \text{on } \partial\Omega. \end{cases}$$

It is conjectured that if there exists a non-trivial solution to the above problem then Ω must be a ball. A recent result by A. Berger [5] says that in two dimensions, the only positive integers k for which the ball is a local minimizer for λ_k under volume constraint are $k = 1, 3$. Finding an optimizer for the problem (1.1) which has simple k-th eigenvalue would prove the above conjecture to be false. On the other hand, in the case of the perimeter constraint, we can find shapes Ω, which are not disks, such that the overdetermined problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = \mathcal{H} & \text{on } \partial\Omega. \end{cases}$$

has a non-trivial solution. Such examples are the shape described in [9] as well as the shapes we found numerically for $k = 6, 9, 13, 15$.

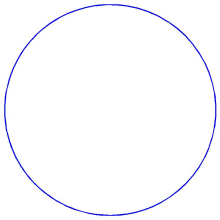
We notice that the numerical optimal shape obtained for $k = 3$ is the disk. This is to be expected, since it is a direct consequence of the conjecture that the ball minimizes $\lambda_3(\Omega)$ under volume constraint. This is still an open problem. A partial result was given by A. Berger which proved that the disk can only be a local minimizer for $k = 1$ or $k = 3$.

We observe that all the optimal shapes computed are symmetric, while this is not the case for the volume constraint where the optimal shape for $k = 13$ is suspected to be non-symmetric [2].

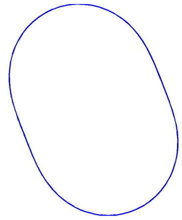
The fact that we can immediately generalize the method in three dimensions is a big advantage. One drawback is the fact that we were not able to obtain very high resolution due to the fact that the matrices involved have extremely large dimensions. The shapes presented in Figure 2 were obtained using a $40 \times 40 \times 40$ grid on $D = [0, 1]^3$. As previously, the initial shape was concentrated around the center of the cube D. In the paper [9] a few conjectures were stated regarding the minimizers in higher dimensions. The first conjecture was that the optimal shape for $\lambda_2(\Omega) + \text{Per}(\Omega)$ is not convex in the three dimensional case. This can be observed in our results. The second conjecture was that the optimal shapes may have cylindrical symmetry. Our numerical computations show that some of the optimizers do not seem to have cylindrical symmetry. Still, some of the optimal shapes seem to be invariant under certain rotations. We notice that the numerical optimal shape for $k = 4$ is the ball. This is a direct consequence of the conjecture that the ball minimizes $\lambda_4(\Omega)$ under volume constraint in three dimensions. We provide for each shape the value of the scale invariant expression $\lambda_k(\Omega)\,\text{Per}(\Omega)$, calculated using a finite element method.

5. Optimality conditions and qualitative results

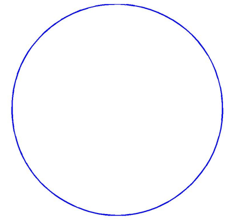
Once we know that a shape optimization problem has a solution, we would like to write some optimality conditions which could allow us to find further qualitative properties. An eigenvalue of the Dirichlet Laplacian associated to a shape Ω is differentiable with respect to perturbations only if it is simple. Unfortunately, solutions of (1.1) and (1.2) are conjectured to have multiple k-th eigenvalue at the optimum (with a few exceptions when we have the perimeter constraint). Thus, classical optimality conditions, like the one exploited in [9], cannot be written for every k. Nevertheless, it is possible to use methods inspired by [14], [13] and [19] in order to overcome the non-differentiability. In the previously cited article [14], the authors provided an optimality condition for problem (1.1), which works even when the eigenvalue is multiple at the optimum. The results of this section are dedicated to finding a similar optimality condition for problem (1.2).



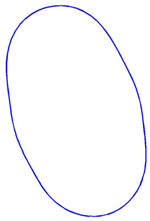
$\lambda_1 = 11.5505$



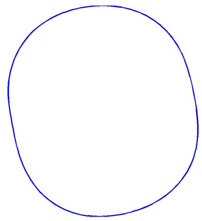
$\lambda_2 = 15.2806$



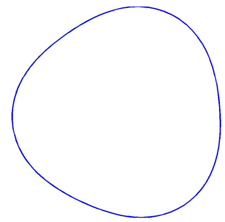
$\lambda_3 = 15.7573$
(double: $\lambda_2 = \lambda_3$)



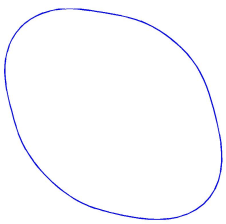
$\lambda_4 = 18.3496$
(double: $\lambda_3 = \lambda_4$)



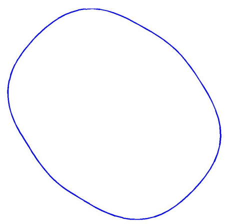
$\lambda_5 = 19.1091$
(double: $\lambda_4 = \lambda_5$)



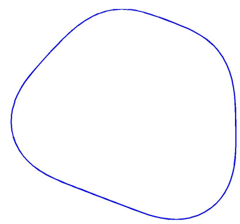
$\lambda_6 = 20.0909$



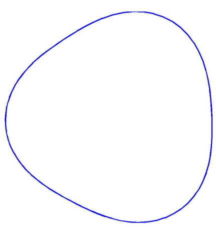
$\lambda_7 = 21.5020$
(double: $\lambda_6 = \lambda_7$)



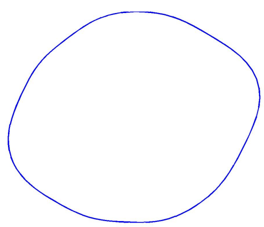
$\lambda_8 = 22.0265$
(double: $\lambda_7 = \lambda_8$)



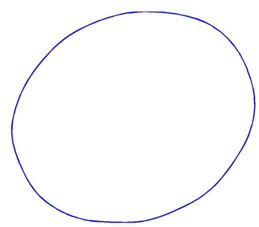
$\lambda_9 = 23.2073$



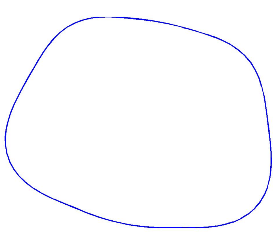
$\lambda_{10} = 23.5501$
(double: $\lambda_9 = \lambda_{10}$)



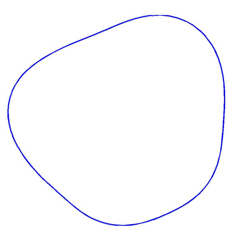
$\lambda_{11} = 24.5970$
(double: $\lambda_{10} = \lambda_{11}$)



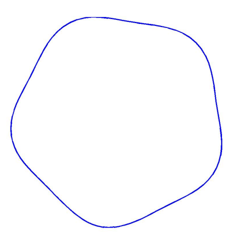
$\lambda_{12} = 24.7440$
(triple: $\lambda_{10} = \lambda_{11} = \lambda_{12}$)



$\lambda_{13} = 25.9823$



$\lambda_{14} = 26.4334$
(double: $\lambda_{13} = \lambda_{14}$)



$\lambda_{15} = 26.9123$

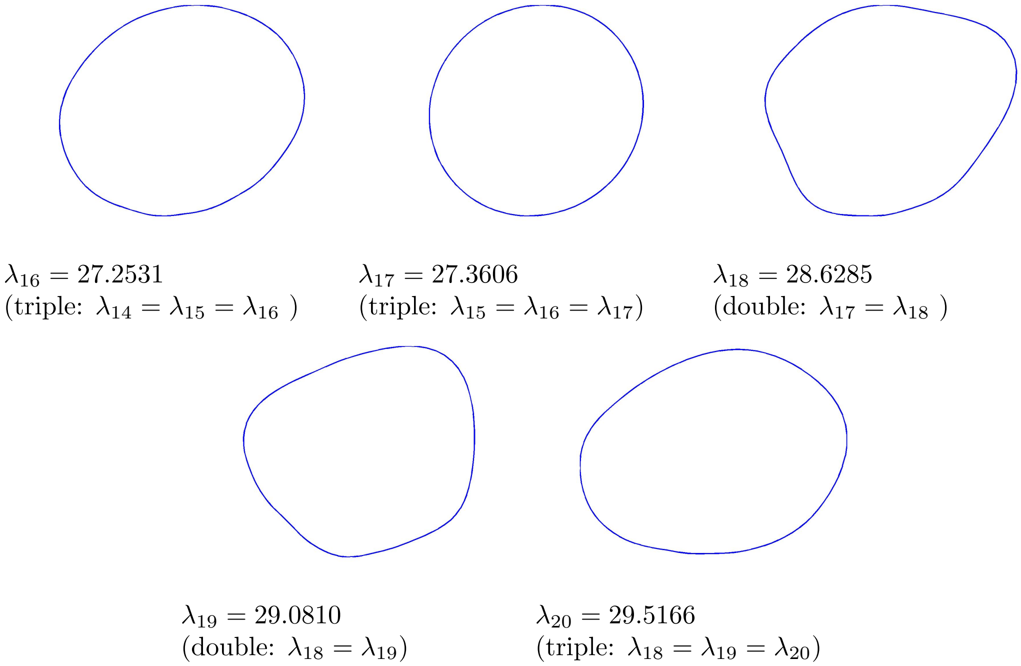


FIGURE 1. Numerical optimizers for problem 1.3 in 2D

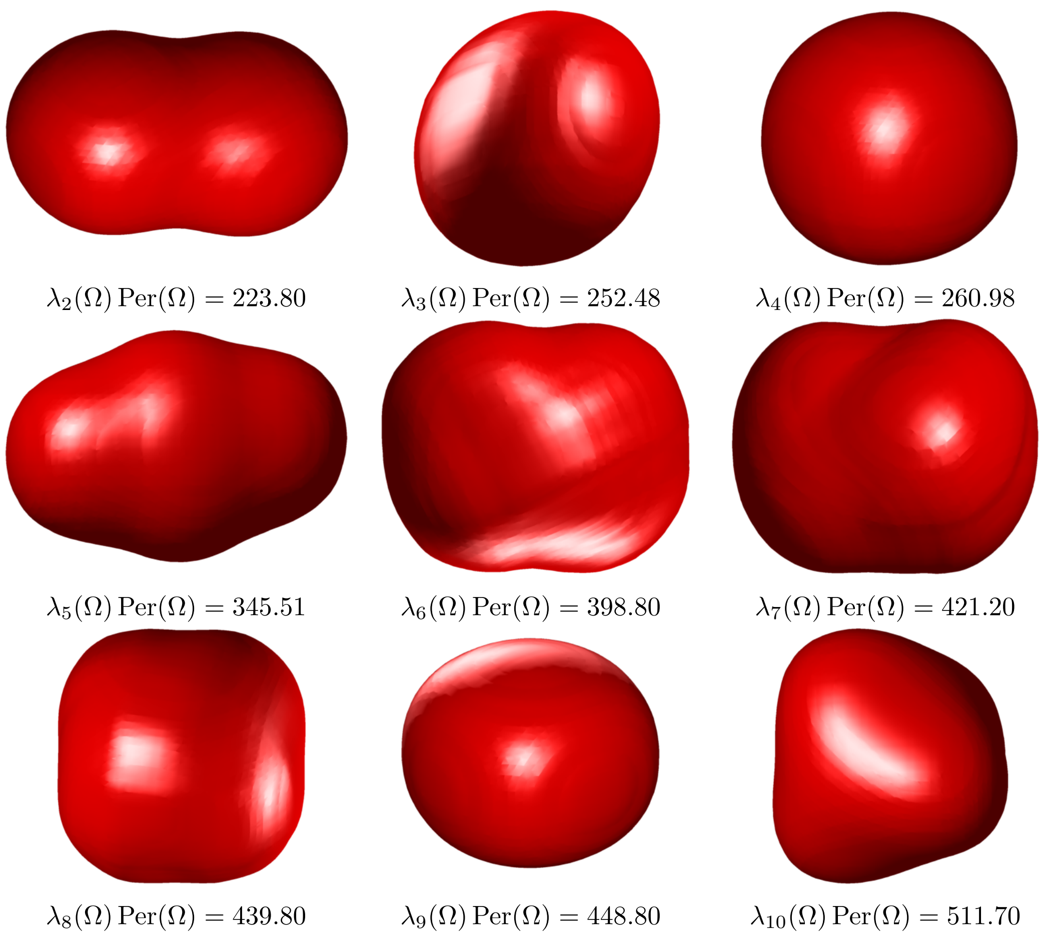


FIGURE 2. Numerical optimizers for problem 1.3 in 3D

k	mult.	Γ-conv	Fourier
1	1	11.55	11.55
2	1	15.28	15.28
3	2	15.75	15.75
4	2	18.36	18.35
5	2	19.11	19.11
6	1	20.09	20.09
7	2	21.50	21.50
8	2	22.07	22.02
9	1	23.21	23.21
10	2	23.58	23.55
11	2	24.64	24.60
12	3	24.76	24.74
13	1	26.02	25.98
14	2	26.50	26.43
15	1	26.92	26.91

TABLE 1. Comparative results

The following theorem is a result similar to Theorem 2.5.10 in [16] where it is said that if an optimizer Ω^* for problem (1.1) is such that the k-th eigenvalue is multiple, then the multiplicity cluster ends at λ_k, i.e. $\lambda_k(\Omega^*) < \lambda_{k+1}(\Omega^*)$. Throughout this section we will assume that Ω has boundary of class C^3. In particular, this implies that its curvature, $\mathcal{H}$ is of class C^1.

Theorem 5.1. *Let $k \geq 2$ such that $\lambda_k > \lambda_{k-1}$ and assume that Ω is a minimizer for the k-th eigenvalue of the Dirichlet Laplacian with a perimeter constraint (i.e. a solution of the problem* (1.2)*). Then λ_k is simple and there exists a unique (up to sign) eigenfunction ϕ satisfying*

$$\begin{cases} -\Delta\phi = \lambda_k\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \\ \left(\frac{\partial\phi}{\partial n}\right)^2 = \mathcal{H} & \text{on } \partial\Omega \end{cases}$$

Proof: Let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a perimeter preserving analytic deformation of Ω and denote $(\Lambda_{i,\varepsilon})_{i\leq p}$ and $(\phi_{i,\varepsilon})_{i\leq p}$ the families of eigenfunctions and eigenvectors associated to λ_k accodrding to Lemma 2.6. Since $\lambda_k = \lambda_{i,0} > \lambda_{k-1}$, by continuity, for sufficiently small ε we have

$$\Lambda_{i,\varepsilon} > \lambda_{k-1,\varepsilon}.$$

We know that Ω is a local minimizer for the Dirichlet Laplacian under the considered perturbation, which means that

$$\Lambda_{i,\varepsilon} \geq \lambda_{k,\varepsilon}.$$

The differentiable function $\varepsilon \mapsto \Lambda_{i,\varepsilon}$ acheives a local minimum at $\varepsilon = 0$ and this implies $\frac{d}{d\varepsilon}\Lambda_{i,\varepsilon} = 0$.

As a consequence, the quadratic form q_v defined in Lemma 2.7 is identically zero on E_k, where $v = \langle \frac{d}{d\varepsilon} f_\varepsilon, n\rangle$. The perimeter preserving deformation is arbitrary, so by Lemma 2.5 we have that q_v vanishes on E_k for every $v \in P_0(\partial\Omega)$. This means that

$$\int_{\partial\Omega} \left(\frac{\partial\phi}{\partial n}\right)^2 v d\sigma = 0$$

for every $v \in P_0(\partial\Omega)$ and for every $\phi \in E_k$.

Hence, for every function $\phi \in E_k$ there exists a constant c such that $\left(\frac{\partial \phi}{\partial n}\right)^2 = c\mathcal{H}$ on $\partial\Omega$. In [12] it is proved that a local minimizer of λ_k for the perimeter constraint has positive mean curvature, so there exists a constant $c_1 = \pm\sqrt{c}$ such that $\frac{\partial \phi}{\partial n} = c_1\sqrt{\mathcal{H}}$ on $\partial\Omega$. If we have two eigenfunctions ϕ_1, ϕ_2 then there exists a linear combination $\phi = \alpha\phi_1 + \beta\phi_2$ such that ϕ vanishes on $\partial\Omega$. We apply Holmgren uniqueness theorem to conclude that $\phi = 0$ and λ_k is simple. □

The following result connects the criticality of a domain Ω with the definiteness of the quadratic form q_v. This will allow us later to state our optimality result.

Theorem 5.2. *Let k be any natural integer.*

1. *If Ω is a critical domain for the k-th eigenvalue of the Dirichlet Laplacian, then, for all $v \in \mathcal{P}_0(\partial\Omega)$, the quadratic form $q_v(\phi) = -\displaystyle\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 v \, d\sigma$ is not definite on E_k.*
2. *Assume that $\lambda_k > \lambda_{k-1}$ or $\lambda_k < \lambda_{k+1}$, and that for all $v \in \mathcal{P}_0(\partial\Omega)$, the quadratic form $q_v(\phi) = -\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 v \, d\sigma$ is not definite on E_k. Then Ω is a critical domain for the k-th eigenvalue of the Dirichlet Laplacian.*

Proof: (1) Consider a function $v \in \mathcal{P}_0(\partial\Omega)$ and let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be an analytic perimeter preserving deformation of Ω such that $v = \langle \frac{d}{d\varepsilon} f_\varepsilon|_{\varepsilon=0}, n\rangle$ (such a deformation exists by Lemma 2.5). Let $(\Lambda_{i,\varepsilon})_{i\le p}$ and $(\phi_{i,\varepsilon})_{i\le p}$ be families of eigenvalues and eigenfunctions associated to λ_k like in Lemma 2.6. There exist two integers $i, j \le p$ such that $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}|_{\varepsilon=0^-} = \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}|_{\varepsilon=0}$ and $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}|_{\varepsilon=0^+} = \frac{d}{d\varepsilon}\Lambda_{j,\varepsilon}|_{\varepsilon=0}$. The criticality of Ω implies that $\frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}|_{\varepsilon=0} \times \frac{d}{d\varepsilon}\Lambda_{j,\varepsilon}|_{\varepsilon=0} \le 0$ and from Lemma 2.7, it follows that q_v has both positive and negative eigenvalues, which means that q_v is not definite on E_k.

(2) Assume $\lambda_k > \lambda_{k-1}$ and let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a volume-preserving deformation of Ω. Let $(\Lambda_{i,\varepsilon})_{i\le p}$ and $(\phi_{i,\varepsilon})_{i\le p}$ be families of eigenvalues and eigenfunctions associated to λ_k according to Lemma 2.6. For ε sufficiently small we have $\lambda_{k,\varepsilon} = \min_{i\le p} \Lambda_{i,\varepsilon}$. Hence

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} = \min_{i\le p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}$$

and

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} = \max_{i\le p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}.$$

The non definiteness of q_v on E_k means that its smallest eigenvallue is non positive and its largest one is non negative. This implies that

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} = \min_{i\le p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0} \le 0$$

and

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} = \max_{i\le p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0} \ge 0$$

which in turn implies the criticality of the domain Ω.

The case $\lambda_k < \lambda_{k+1}$ can be treated in a similar manner. □

The next result provides a nice characterisation of the non-definitness of q_v. Note that unlike in [14], we had to add some hypothesis on $\mathcal{H}$, but this hypothesis is natural when dealing with solutions of problem (1.2).(see [12], Section 4.)

Theorem 5.3. *Let k be a natural integer. If Ω is such that its curvature satisfies $\mathcal{H} \ge 0$ and $\int_{\partial\Omega} \mathcal{H} \ne 0$ then the following two conditions are equivalent:*

(i) *For all $v \in \mathcal{P}_0(\partial\Omega)$, the quadratic form q_v is not definite on E_k.*

(ii) *There exists a finite family of eigenfunctions* $(\phi_i)_{i\leq m} \subset E_k$ *satisfying*

$$\sum_{i=1}^{m} \left(\frac{\partial \phi_i}{\partial n}\right)^2 = \mathcal{H} \text{ on } \partial\Omega.$$

Proof: To see that (ii) implies (i) it suffices to notice that, for any $v \in \mathcal{P}_0(\partial\Omega)$

$$\sum_{i\leq m} q_v(\phi_i) = -\sum_{i\leq m} \int_{\partial\Omega} \left(\frac{\partial \phi_i}{\partial n}\right)^2 v d\sigma = -\int_{\partial\Omega} \mathcal{H}\ v\ d\sigma = 0,$$

which means that q_v is not definite on E_k.

To prove the other implication we look at $K = \text{conv}\{\left(\frac{\partial\phi}{\partial n}\right)^2,\ \phi \in E_k\}$, and we want to prove that the function $\mathcal{H}$ belongs to K. Suppose that $\mathcal{H} \notin K$. Then, from the Hahn-Banach theorem (applied to the finite dimensional normed vector subspace of $C^1(\partial\Omega)$ spanned by K and $\mathcal{H}$), there exists a function $v \in C^1(\partial\Omega)$ such that $\int_{\partial\Omega} \mathcal{H}\ v\ d\sigma > 0$ and for all $\phi \in E_k$,

$$\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 v\ d\sigma \leq 0.$$

Since v is not necessarily in $\mathcal{P}_0(\partial\Omega)$, we modify it by a constant term and define $v_0 = v - c$ where c is chosen such that $v_0 \in \mathcal{P}_0(\partial\Omega)$. The condition that c must satisfy is

$$0 = \int_{\partial\Omega} \mathcal{H}\ v_0\ d\sigma = \int_{\partial\Omega} \mathcal{H}\ v\ d\sigma - c\int_{\partial\Omega} \mathcal{H}\ d\sigma.$$

This last relation defines $c = \dfrac{\int_{\partial\Omega} \mathcal{H}\ v}{\int_{\partial\Omega} \mathcal{H}\ d\sigma}$ provided that $\int_{\partial\Omega} \mathcal{H}\ d\sigma \neq 0$. Furthermore, regarding the hypothesis we have on v and $\mathcal{H}$ we see that we have $\int_{\partial\Omega} \mathcal{H}\ v\ d\sigma > 0$ and $\int_{\partial\Omega} \mathcal{H}\ d\sigma > 0$, which implies $c > 0$.

For $\phi \in E_k$ we have

$$\begin{aligned} q_{v_0}(\phi) &= -\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 v_0\ d\sigma \\ &= -\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 v\ d\sigma + c\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 d\sigma \\ &\geq c\int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n}\right)^2 d\sigma \end{aligned}$$

and $\int_{\partial\Omega} \left(\frac{\partial\phi}{\partial n}\right)^2 d\sigma > 0$ for any non trivial Dirichlet eigenfunction ϕ (due to Holmgren uniqueness theorem). In conclusion, we have found a function $v_0 \in \mathcal{P}_0(\partial\Omega)$ such that the quadratic form q_{v_0} is positive definite on E_k, which contradicts condition (i). $\square$

Corollary 5.4. *If* Ω *is a local minimizer for the problem* (1.2)

$$\min_{\text{Per}(\Omega)=1} \lambda_k(\Omega)$$

with boundary of class C^3*, then there exists a finite family of eigenfunctions* $(\phi_i)_{i\leq p} \subset E_k$*, such that*

$$\sum_{i=1}^{m} \left(\frac{\partial \phi_i}{\partial n}\right)^2 = \mathcal{H}.$$

Proof: It is a direct result of the above theorems, noting that any solution Ω of the considered problem must verify $\mathcal{H} \geq 0$ [12]. Furthermore, we must have $\int_{\partial\Omega} \mathcal{H} d\sigma > 0$, since equality would imply $\mathcal{H} = 0$ everywhere, which is a contradiction, since Ω is open and cannot be flat. $\square$

In the article [9] the authors prove that the solution of (1.2) in the case $k = 2, d = 2$ has no segments and no arcs of circles in its boundary. The method used in the mentioned article works only in the case we know the corresponding eigenvalue is simple. Using the above corollary, we can partially extend this result to the general case. In the following, we call a *flat part* of $\mathbb{R}^d$, the intersection of a $d-1$ dimensional hyperplane with a d-dimensional ball.

Theorem 5.5. *If Ω is a local minimizer for the problem* 1.2

$$\min_{\text{Per}(\Omega)=1} \lambda_k(\Omega)$$

then $\partial\Omega$ does not contain a flat parts.

Proof: Suppose that Ω contains a flat part S in its boundary. Using the previous convention, $S = H \cap B$ where H is a $d-1$ dimensional hyperplane and B is a d-dimensional ball. Then $\mathcal{H} = 0$ on that region S, and by Corollary 5.4, at least one eigenfunction ϕ satisfies $\dfrac{\partial \phi}{\partial n} = 0$ on that S.

We then choose an extension $\Omega' = \Omega \cup B'$ of the domain Ω such that B' is a ball, $B' \subset B$, $B' \not\subset \Omega$ and B' is small enough such that $B' \cap \partial\Omega \subset S$. Define $\phi' = \phi$ on Ω and 0 on $\Omega' \setminus \Omega$. This will create an an eigenfunction ϕ' on Ω' which is zero on an open set. This together with the analiticity of ϕ' and the fact that ϕ' is not identically zero brings us to a contradiction.

In conclusion, Ω cannot contain a flat part in its boundary. □

5.1. **Numerical computation of the optimality conditions.** By the above results, we know that if Ω is a minimizer for (1.2) then it exists a family of eigenfunctions $(\phi_i)_{i=1}^m \subset E_k$ such that

$$\sum_{i=1}^m \left(\frac{\partial \phi_i}{\partial n}\right)^2 = \mathcal{H}. \tag{5.1}$$

In order to evaluate the numerical quality of our solutions we would like to investigate how far our solutions satisfy this optimality condition. The question is whether we are able to find a combination of eigenfunctions which realize this equality. Suppose that $\dim E_k = p$ and the p orthonormal eigenfunctions which span E_k are denoted $u_1, ..., u_p$. It is easy to see that (5.1) implies that

$$\mathcal{H} \in \text{span}\left(\left\{ \left(\frac{\partial u_i}{\partial n}\right)^2,\ i = 1..p \right\} \bigcup \left\{ \frac{\partial u_i}{\partial n}\frac{\partial u_j}{\partial n},\ 1 \leq i < j \leq p \right\} \right).$$

This observation is a direct consequence of the fact that each ϕ_i can be written as

$$\phi_i = \sum_{j=1}^p \alpha_j^i u_j.$$

Thus, in a first step, we can find the coefficients of $\mathcal{H}$ in the decomposition

$$\mathcal{H} = \sum_{i=1}^p \alpha_i \left(\frac{\partial u_i}{\partial n}\right)^2 + \sum_{1 \leq i < j \leq p} \beta_{i,j} \frac{\partial u_i}{\partial n}\frac{\partial u_j}{\partial n}$$

by solving an optimization problem. The normal derivatives $\frac{\partial u_i}{\partial n}$ and the curvature are known on a discretization $\{x_1, ..., x_l\}$ of the boundary $\partial\Omega$. To find the coefficients, we solve the quadratic, convex minimization problem

$$\min_{\substack{(\alpha_i)_{i=1}^p,\\ (\beta_{i,j})_{1\leq i<j\leq n}}} \sum_{h=1}^l \left(\sum_{i=1}^p \alpha_i \left(\frac{\partial u_i}{\partial n}(x_h)\right)^2 + \sum_{1\leq i<j\leq p} \beta_{i,j} \frac{\partial u_i}{\partial n}(x_h)\frac{\partial u_j}{\partial n}(x_h) - \mathcal{H}(x_h) \right)^2$$

k	mult.	Combinations which realize the optimality conditions
1	1	-
2	1	-
3	2	$(\frac{1}{\sqrt{2}}\frac{\partial u_2}{\partial n})^2+(\frac{1}{\sqrt{2}}\frac{\partial u_3}{\partial n})^2=\mathcal{H}$
4	2	$(0.16\frac{\partial u_3}{\partial n}-0.06\frac{\partial u_4}{\partial n})^2+(0.98\frac{\partial u_4}{\partial n})^2=\mathcal{H}$
5	2	$(0.54\frac{\partial u_4}{\partial n}-0.02\frac{\partial u_5}{\partial n})^2+(0.83\frac{\partial u_5}{\partial n})^2=\mathcal{H}$
6	1	-
7	2	$(0.70\frac{\partial u_6}{\partial n}-0.37\frac{\partial u_7}{\partial n})^2+(0.59\frac{\partial u_7}{\partial n})^2=\mathcal{H}$
8	2	$(0.39\frac{\partial u_7}{\partial n}-0.02\frac{\partial u_8}{\partial n})^2+(0.92\frac{\partial u_8}{\partial n})^2=\mathcal{H}$
9	1	-
10	2	$(0.70\frac{\partial u_9}{\partial n})^2+(0.70\frac{\partial u_{10}}{\partial n})^2=\mathcal{H}$
11	2	$(0.85\frac{\partial u_{10}}{\partial n}-0.05\frac{\partial u_{11}}{\partial n})^2+(0.51\frac{\partial u_{11}}{\partial n})^2=\mathcal{H}$
12	3	$(0.4619\frac{\partial u_{10}}{\partial n}-0.1423\frac{\partial u_{11}}{\partial n}-0.2617\frac{\partial u_{12}}{\partial n})^2+$ $(0.4137\frac{\partial u_{11}}{\partial n}+0.2728\frac{\partial u_{12}}{\partial n})^2+(0.6715\frac{\partial u_{12}}{\partial n})^2=\mathcal{H}$
13	1	-
14	2	$(0.56\frac{\partial u_{13}}{\partial n})^2+(0.82\frac{\partial u_{14}}{\partial n})^2=\mathcal{H}$
15	1	-

TABLE 2. Optimality conditions in two dimensions

Then, we transform this quadratic representation into a canonical representation by using the classical Gauss-Jacobi method. Of course, this representation is not unique. The claim of Corollary 5.4 is that this canonical representation will consist in a sum of squares: to test this, we checked if the matrix $(a_{i,j})$ defined by $a_{i,i}=\alpha_i,\ a_{i,j}=a_{j,i}=\beta_{i,j}/2$ is positive definite. The answer is affirmative for every optimizer, and a representation of the type (5.1) is presented for each $k=1,...,20$ in Table 2. In all computations we check the pointwise optimality conditions presented in Table 2 up to an upper bound of 0.03.

The shape Ω_{15}, we obtained for $k=15$ seemed to have triple eigenvalue. We have obtained that $\sqrt{\lambda_{15}(\Omega_{15})}=2.9951$ and $\sqrt{\lambda_1 3(\Omega_{15})}=\sqrt{\lambda_1 4(\Omega_{15})}=2.9900$. On the other hand, the optimality condition satisfied by Ω_{15} is the same as for a simple eigenvalue, i.e. $\left(\frac{\partial u_{15}}{\partial n}\right)^2=c\mathcal{H}$. Thus, we are inclined to believe that $\lambda_{15}(\Omega_{15})$ should be simple.

Finally, we have observed that the optimality condition is a strong indicator of a local minimum. At first, when we verified if the optimality condition is satisfied on the results we obtained, we got large errors. We then decided to remake the initial computations and it turned out that in every situation where the optimality error was large, we were able to go further with the optimization and decrease even more the optimal value.

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